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Developmental Change and Individual Differences in Children's Multiplication

by

Donald James Mabbott ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfilment of
the requirements for the degree of Doctor of Philosophy.

Department of Psychology

Edmonton, Alberta

Spring, 1998

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled Developmental Change and Individual Differences in Children's Multiplication submitted by Donald James Mabbott in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Dr. Jeffrey Bisanz (Supervisor)

Connie K. Varnhagen
Dr. Connie Varnhagen (Chair)

Dr. Gay L. Bisanz

Dr. Tom Kieran

Dr. Lori Buchanan

Connie K. Varnhagen
Dr. Mark Ashcraft (External)

Date: 7 January 1998

Abstract

The goal of this paper is to provide an account of the development of multiplication skill and knowledge. I examined age-related changes and individual differences for Grade 4 and 6 children in (a) the computational skills used to solve simple multiplication problems including the procedures used and indices thought to reflect the representation of facts in memory, (b) concepts important for understanding multiplication, and (c) working memory important for mathematics. The use of multiple measures allowed for the opportunity to provide an integrated account of the development of multiplication including evaluating the developing relations between different areas important for multiplication. The performance of 60 children from Grade 4 and 60 children from Grade 6 was evaluated using a number of tasks. Specifically, children solved 28 multiplication problems. Accuracy and latency were recorded, as were immediately retrospective self-reports on how the problem was solved. As well, children solved a number of problems based on the following concepts: (a) commutativity, (b) relations between repeated addition and multiplication, (c) part-whole relations, (d) relative effects of operations on numbers, (e) relative magnitudes of numbers, (f) concepts important for solving word problems, and (g) relations between concrete manipulatives and symbolic representations of specific problems. Working memory was assessed using a backward digit span and an operation span task. Finally, children's performance on a test of mathematical achievement was evaluated. I terms of simple multiplication I found that: children use multiple procedures to solve multiplication problems, problem characteristics are important in predicting solution latencies, children use specific procedures on specific types of problems, and variables important for predicting solution latencies change with age. I found that conceptual understanding in multiplication is influenced by the context in which it is assessed, and that the development of concepts is uneven. Specifically, it appears that concepts which can be directly applied to solve problems are acquired first, followed by concepts used to enable solving novel problems. In terms of working memory, it appears that mathematical achievement is related to domain-specific working memory in Grade 4 but global working memory in Grade 6. Implication of the results are discussed in terms the appropriate assessment of children's performance, models of performance in multiplication, principles of development, and the development course in the acquisition of multiplication skill and knowledge.

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DEVELOPMENTAL CHANGE AND INDIVIDUAL DIFFERENCES IN CHILDREN'S MULTIPLICATION

Human cognition is characterized by change. Memory changes as new information is added (e.g., Loftus, 1992) and young children often solve problems differently than older children, adolescents, and adults (e.g., Inhelder & Piaget, 1958; Piaget, 1952). Researchers interested in human thinking must try to understand the changes that occur with development and mathematics is a useful domain for studying these changes.

The importance of mathematics is evident across the developmental spectrum. Young children must make decisions regarding "more" or "less" and learn to count to interact with their environment. Mathematics is an important part of the curriculum for school-aged children. As well, the acquisition of mathematical skill is necessary to function in a complex financial and technological society such as ours. Sales clerks must use basic mathematics in retail transactions. Individuals in the construction industry must understand measurement and use geometric principles. Physicians and nurses require mathematics ability to calculate an appropriate dosage of medicine. Accountants, engineers, and scientists all use mathematics. Research on the processes that underlie mathematics has been important for generating insights about the development of remembering, problem-solving, and conceptual understanding, and these insights have led to hypotheses and conclusions that extend beyond the domain of arithmetic.

Unfortunately, an integrated account of development in mathematics has not been provided because most researchers have focused exclusively on a single area of cognition:

either computational skills, conceptual knowledge, or working memory. Although it is important to study and understand change within a specific area, an unfortunate consequence of this approach is that research on mathematics has become fragmented. To address the fundamental questions about the acquisition of mathematics skill an integrated account of the different areas of cognition is needed.

My goal is to provide an account of age-related changes in children's multiplication skill and knowledge that includes examination of computational skills, conceptual knowledge, and working memory. The paper is organized using two themes: Developmental Changes in Multiplication Skill and Knowledge; and Individual Differences and Mathematical Achievement. It is important to evaluate age-related changes to understand the acquisition of multiplication skill and knowledge. Although normative models may fit group data well, they may misrepresent the patterns in data from each individual in the group (Widaman & Little, 1992). Consequently, it is also important to examine individual differences, and relations among measures of mathematics skill.

The Introduction is divided into three sections. In the first section the importance of multiplication skill and knowledge is discussed and a rationale for the use of multiple measures is presented. To study changes in multiplication skill, clearly defined models are needed in order to measure important aspects of behavior. Consequently, the underlying theories of computational skill, conceptual knowledge, and working memory are also presented. In the second and third sections, background information and rationale are provided for the study of (a) developmental changes, and (b) individual differences in multiplication skill and knowledge. In each of these sections, previous research related to computational skill, conceptual knowledge and working memory is reviewed. As well, the empirical questions addressed in this study, possible outcomes, and implications for development are presented.

The organization of the Results and Discussion section follows that of the Introduction. First, data related to age-related changes in multiplication skill and knowledge are presented. Second, data regarding individual differences in multiplication skill and knowledge, and mathematical achievement are presented. Finally, conclusions regarding the development of multiplication are reviewed in the General Discussion section.

Models of Multiplication Skill and Knowledge

Children's acquisition of basic multiplication knowledge is important for complex mathematical processing. As children encounter advanced mathematics problems (i.e., multi-digit multiplication), the ability to compute multiplication facts efficiently allows them to focus on the problem-solving strategies required for accurate performance. Children with poor multiplication skills often have difficulty in advanced mathematics. Although researchers have evaluated mathematical skills important for children's counting, addition, subtraction (Siegler & Schrager, 1984), and for addition and multiplication in adults (Ashcraft, 1987; Widaman & Little, 1992; LeFevre, Bisanz, Daley, Buffone, Greenham, & Sadesky, 1996; LeFevre, Sadesky, & Bisanz, 1996), few investigations have been conducted on processes that underlie the development of multiplication in children

(Ashcraft & Christy, 1995; Campbell & Graham, 1985; Cooney, Swanson, & Ladd, 1988; Koshmider & Ashcraft, 1991; Siegler, 1988a).

In this study multiple measures of computational skills, conceptual knowledge, working memory, and mathematical achievement were used to examine age-related changes, and individual differences in multiplication abilities of children in Grades 4 and 6. The use of multiple measures allowed the opportunity to provide an integrated account of the development of multiplication not present in the existing literature. By using measures of computational skills, conceptual knowledge, and working memory, I was able to identify some principles of mathematical development that are common to all areas. As well, having a number of measures on each subject was useful in evaluating the developing relations between different areas important for multiplication.

Computational Skill

In most models of arithmetic performance, computational skill is considered to consist of (a) the representation of number facts in long-term memory, and (b) the procedures children use to access those facts. Procedures are mental operations or sequences of operations that occur over time, accomplish a goal, and can be stored in memory (Bisanz & LeFevre, 1990). Studying computational skill in multiplication provides information regarding changes in the representation of mathematical knowledge and the different mental operations children use to solve problems. Two families of models have been used to characterize children's computational performance in multiplication. Problem-answer association models are based primarily on direct retrieval of problem-answer associations from memory. Memory/procedure models are similar; but they also incorporate alternative procedures children use when solving problems.

Problem-Answer Association Models

Most models of arithmetic performance in children and adults share the assumption that numbers are stored as nodes in an associative network in long-term memory (Ashcraft, 1982, 1992; Campbell & Graham, 1985; Campbell, 1987). When an arithmetic problem is presented, the network becomes activated and the activation spreads to the nodes representing the numbers specified in the problem. These models differ in how the candidate nodes become activated and how the network is organized, and so the ways in which they are used to explain empirical findings differ as well. The most frequent empirical finding in simple arithmetic is the problem-size effect: as the magnitudes of the operands in a single digit arithmetic problem increase, so do solution times and errors (Ashcraft & Christy, 1995; Campbell & Graham, 1985; Cooney et al., 1988; Koshmider & Ashcraft, 1991). For example, latencies for "large" problems, such as 7 x 8, tend to be longer than for "small" problems, such as 2 x 3.

In structural accounts of arithmetic performance, problems are thought to be stored in a representation similar to a two-dimensional table (Ashcraft & Battaglia, 1978; Widaman & Little, 1992; Widaman, Geary, Cormier, & Little, 1989). Operands from 0 to 9 define the rows and columns of the table, and correct answers are stored at the intersection of the rows and columns. When a problem is presented, activation spreads from the corresponding operands across the table until an intersection occurs at the location of the product (Widaman et al., 1989). Time to retrieve an answer is assumed to be the time required for the search across the table until the row and column intersect, which in turn is a function of the size of the operands. Because problems with large operands require searching through a greater number of links than problems with small operands, the problem-size effect is the result of the search distance in the network. A number of structural variables have been used as indices of the problem-size effect, including sum of the operands (Stazyk, Ashcraft, & Hamann, 1982), sum square (Ashcraft & Battaglia, 1978), and the product of operands (Miller, Perlmutter, & Keating, 1984). Although this line of research has led to insights about how arithmetic facts may be represented in memory, relatively little attention has been paid to the process through which facts are acquired or to how the representation of number facts may change with development.

In learning or experiential models, arithmetic facts are stored in a network with nodes representing problem operands connected to nodes representing answers. The strength of associations between nodes varies as a function of the experience an individual has during the learning of basic multiplication facts. Acquisition of number facts and age-related differences in performance are linked to frequency of exposure and order of acquisition (Ashcraft, 1987; Ashcraft & Christy, 1995; Campbell & Graham, 1985; Campbell, 1987; Graham & Campbell, 1992). Because small-number problems are encountered sooner and more often than large-number problems, the predictive power of structural variables derives, in part, from a correlation with order of acquisition and frequency (Campbell & Graham, 1985). By this account, the problem-size effect is not a consequence of the distance searched through the network. Instead, it is largely the result of large-number problems being tested less often and occurring later in the learning sequence. However, frequency alone cannot account for the advantages shown by some classes of problems. For example, tie problems and problems with "5" as an operand are not presented more frequently than other problems of a similar magnitude, but children solve them more quickly and accurately (LeFevre et al., 1996a).

Memory/Procedure Models

In memory/procedure models of multiplication, number facts are considered to be stored in an associative network, but retrieval from that network is assumed to be supplemented by other procedures. Siegler and Shipley (1995) developed the Adaptive Strategy Choice Model (ASCM) to account for how children choose a procedure and progress from using counting-based procedures to use of retrieval. In ASCM each fact is associated with a variety of possible answers. The strongest or most "peaked" association develops between the problem and its correct solution, but related answers may also have appreciable connections, resulting in a distribution of associations. ASCM also includes stored procedural knowledge about non-retrieval solutions. Whether retrieval or some other procedure is chosen for problem solving is governed by the relative strengths of all associated procedures and associative strengths in the stored representation of answers.

When a problem is solved, ASCM stores information not only about the answer to the particular problem but also about the solution procedure. This information is then used to modify the data base regarding the procedure, the problem, and their interaction. Knowledge of each procedure includes information about (a) its past speed and accuracy aggregated over all problems (global data), (b) its speed and accuracy on problems with a particular feature (featural data), (c) its speed and accuracy on each particular problem (problem-specific data), and (d) its newness (novelty data). Whenever a problem is presented, speed, accuracy, and novelty data for each procedure are used to make projections concerning how well the procedure is likely to do in solving the problem. Selection of a procedure is based on the projected strength of each procedure as determined by global data, featural data, problem-specific data, and novelty data (the strength of a relatively new procedure is boosted beyond what it's past performance alone would justify). The probability of choosing a particular procedure is proportional to that procedure's projected strength relative to that of all procedures combined. If a backup procedure is chosen, it is executed to completion. If retrieval is chosen an individual attempts to retrieve an answer to a problem directly from his or her knowledge base of arithmetic facts. The probability of any given answer being retrieved on a particular retrieval effort is proportional to the associative strength of that answer relative to the associative strengths of all answers to the problem. If the associative strength of the retrieved answer exceeds a pre-set confidence criterion, then the child states the answer (Siegler & Schrager, 1984). If not, the individual tries to retrieve again or chooses a backup procedure.

Lemaire and Siegler (1995) used ASCM as a basis for making several predictions about developmental changes in the use of procedures. First, use of retrieval should become increasingly frequent with development. Answers become associated with the problem on which they were stated and the increment is greater for correct answers than for incorrect ones. Assuming that backup procedures result in the correct answer more frequently than any other answer, the correct answer grows in strength relative to all other answers. As the associative strength between the problem and the correct answer grows relative to associations between the problem and incorrect answers, the probability of retrieving the correct answer grows. The increasing success of retrieval also leads to it being tried before other procedures. Second, execution of procedures should improve in terms of accuracy and latency. Knowledge about each procedure's past speed and accuracy are stored. Consequently, when children can choose among alternative ways of executing a given procedure, they should increasingly choose the ones that are fastest and that yield the most accurate results. Finally, choice among procedures should become more adaptive. In ASCM, the same factors that determine problem difficulty -the number and difficulty of operations needed to execute backup procedures correctly - also determine probability of use of retrieval. Consequently, the more difficult the problem, the more likely that a backup procedure will be used. As experience in solving problems is gained, ASCM predicts that the ability to estimate the likely effectiveness of each procedure for specific problems should increase.

Conceptual Knowledge

Educators and researchers have often ignored the potentially important role of conceptual knowledge in children's acquisition of multiplication. Children are typically taught to memorize multiplication facts. Most research on multiplication is related to the storage of number facts in memory and the procedures used to access those facts (Ashcraft & Christy, 1995; Campbell & Graham, 1985; Lemaire & Siegler, 1995). Mathematics involves more than rote memorization or learning algorithms, and a complete account of multiplication skill should include evaluation of the underlying concepts.

Researchers have examined conceptual knowledge in areas such as rational numbers (Behr, Lesh, Post, & Silver, 1983), whole numbers and numeration (Steffe & VonGlaserfeld, 1983), strategy use (Dixon & Moore, 1996), and the relations with procedural knowledge (Ohlsson & Rees, 1991). A number of terms have been used to describe conceptual knowledge, including intuitive understanding, informal understanding, and principled knowledge. Despite the diverse domains studied and different terms used, many researchers have consistently defined conceptual knowledge in terms of the underlying principles for a domain and their interrelations (Baroody & Ginsburg, 1986; Byrnes & Wasik, 1991; Dixon & Moore, 1996; Hiebert & LeFevre, 1986).

Concepts Important for Multiplication

When researchers have studied conceptual development in multiplication their focus has been narrow. For example, Nunes and Bryant (1995) examined only commutativity in multiplication. However, many forms of behavior are related to conceptual knowledge (Bisanz & LeFevre, 1992) and understanding multiplication involves "understanding" a host of related concepts that can be assessed in different contexts. These concepts include (a) commutativity, (b) relations between repeated addition and multiplication, (c) part-whole relations, (d) relative effects of operations on numbers, (e) relative magnitudes of numbers, (f) concepts important for solving word problems, and (g) relations between concrete manipulatives and symbolic representations of specific problems. These concepts provide the basis for the flexible application of multiplication skill in problem-solving, not just retrieval of facts from memory.

Commutativity. An important mathematical principle in children's understanding of arithmetic is commutativity of multiplication (i.e., a x b = b x a). Nunes and Bryant (1995) found that commutativity of multiplication was relatively late in developing (10 years of age), and that understanding was influenced by the context in which the problem was presented. Assessment of commutativity often involves examining whether a child can directly apply the principle to solve a problem. For example, Baroody and Gannon (1984) studied kindergarten children's understanding of the principle of commutativity by examining their solutions for a number of problems. Each child was first presented with a problem to solve, such as 6 + 4. Next, the experimenter wrote down the same addends in the reverse order, 4 + 6, and asked if the sum would be 10. If a child understood that addition was commutative, then he would answer "yes" without having to recalculate the answer. A child who did not understand that addition was commutative would add 4 and 6, usually by counting, before answering the question.

Repeated addition. Most elementary school mathematics programs use models such as sets, arrays, number lines, and successive addition to teach multiplication. Central to each of these methods is the procedure of repeated addition of equal addends (Cooney et al., 1988). If children recognize the connection between repeated addition and multiplication, they can directly apply their understanding of repeated addition to solve multiplication problems (e.g., solving 4 x 13 by using 13 + 13 + 13 +13).

Part-whole relations. In multiplication the part-whole schema specifies relations among three numbers, a whole and two parts that multiply to the value of the whole (Resnick, 1983). Interpreting quantities in terms of parts and wholes permits children to think of numbers as compositions of other numbers, enabling them to develop flexible numerical computation (Greeno, 1991). Because procedures such as grouping and repeated addition are often used to teach multiplication, an understanding of part-whole relations is important for developing multiplication skills. Derived-fact strategies have been used to examine children's understanding of part-whole relations. They involve using a set of known number facts to derive the solution to unknown combinations. For example, when solving 4 x 6 children may break the problem into two parts. One part may be a fact that is already known (i.e., 4 x 5 = 20). They then add the remaining 4 to that product to get the answer (i.e., 20 + 4 = 24). Grade 3 and 4 children have been reported to use derived-fact strategies on 20-50% of selected simple addition and subtraction problems (Hubbard, 1994; Putnam, deBettencourt, & Leinhardt, 1990).

Relative effects of operations on numbers: Number sense. Number sense refers to several important but elusive capabilities, including flexible mental computation, numerical estimation, and quantitative judgment (Greeno, 1991; Howden, 1989). It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms (Howden, 1989). Two related principles of number sense important for multiplication are understanding the relative magnitudes of numbers and understanding the relative effects of operations on numbers (Thompson & Rathmell, 1989). The product-less-than-operand principle specifies the allowable minimum magnitude of the product in multiplying two whole numbers (i.e., that the product must not be less than either operand). The product-equal-to-operand principle specifies that the product can only equal an operand if the other operand is one (Krzanowska, 1988). These principles can be assessed by having children directly apply them to evaluate the accuracy of answers for specific problems. Krzanowska (1988) found that 40% of Grade 4 and 6 children use the product-equal-to-the-operand principle and 24% use the product-less-than-operand principle to solve verification problems in multiplication. Number sense also involves the ability to estimate answers to problems. For estimation in multiplication, children's understanding of place-value and the influence of arithmetic operations enables them to generate an answer close to the correct answer.

Word problems. Many children at all ages have difficulty with non-routine problems that require some analysis or thinking (Carpenter, Corbitt, Kepner, Linquist, & Reys, 1980; Carpenter, Matthews, Lindquist, & Silver, 1984). Solving word problems requires a thoughtful analysis: defining the problem; planning a solution strategy; implementing the solution strategy; and checking the results (Baroody, 1987). A thoughtful analysis entails conceptual understanding. Specifically, to solve word problems children must generalize previously learned information to a novel setting. Use of conceptual knowledge helps children in deciding what information is needed to solve the problem (and what information is irrelevant), which solution methods are appropriate (and which are inappropriate), and which solutions are reasonable (and which indicate the need for further effort) (Baroody, 1987).

Proofs using manipulatives. In teaching multiplication concrete manipulatives are often used to demonstrate the symbolic representation of specific problems. For example, to demonstrate that 4 x 3 = 12, a teacher may separate colored blocks into 4 groups of 3 and ask the children to add them together to make 4 groups of 12 blocks. Conceptual knowledge involves seeing the relations between concrete and symbolic information.

Categories of Multiplication Concepts

Based on the task demands and materials used to evaluate understanding, the concepts important for multiplication can be defined in two ways: (a) direct application, and (b) enabling-application of concepts. Direct application of concepts in defined as children's ability to directly apply some concepts to a problem-solving situation. The direct application of concepts has been used as a measure of conceptual knowledge (Baroody & Gannon; 1984). Presumably, children's understanding of these concepts fulfils the requirements or conditions of a specific problem situation, or it does not. If knowledge of the concept meets the problem requirements, then children's existing knowledge is sufficient for solving the problem. For example, if children understand the principle of commutativity, they do not need to recalculate the answer to the problem if the operands are simply reversed. They know that the answer must be the same, because if the operands are the same then their position does not matter.

Enabling application of concepts is defined as children's use of concepts to generate new operations and use them in a flexible manner. Specifically, conceptual knowledge provides a starting point that enables inferences and operations beyond the original concept to be carried out. Typically this enabling function involves the application of previous knowledge to a novel task (Bisanz & LeFevre, 1990; Greeno, Riley, & Gelman, 1984; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989). For example, if children understand part-whole relations, then they can use combinations of known number facts to derive the solution to a novel problem (Putnam et al., 1990).

Working Memory

The concept of working memory is commonly invoked as a mechanism for the processing and temporary storage of information in a wide variety of cognitive tasks (Logie, Gilhooly, & Wynn, 1994). Researchers have claimed that the limited capacity of working memory can place constraints on cognitive tasks such as reasoning, problem-solving, and reading comprehension (Daneman & Carpenter, 1983; Logie et al., 1994; Turner & Engle, 1989). Although few researchers have explicitly examined the role of working memory in relation to multiplication skills, most agree that temporary storage and processing of information is required during calculation of number facts, even single digit multiplication problems (Ashcraft, Donley, Halas, & Vakali, 1992; Logie et al., 1994).

A Model of Working Memory

Baddeley and his colleagues (Baddeley, 1986; Baddeley & Hitch, 1974) have developed a model of working memory that includes three components: a central executive and two slave systems, the visuo-spatial scratch pad and the articulatory loop. The central executive is considered to be a limited capacity system that is involved in cognitive processing, such as problem-solving, and in coordination of the activities of the other two components. Part of the limited capacity of the central executive is thought to be responsible for the processing of information and the remainder is used for the storage of the products resulting from that processing. The visuo-spatial scratch pad and the articulatory loop are maintenance systems controlled by the central executive and they are primarily responsible for the storage of either visual-spatial or verbal information.

Memory Span Tasks

Measurement of working memory capacity typically involves combing two tasks: a processing (primary) task and a disruption or storage (secondary) task. In memory span tasks, subjects are required to work on a primary task and at the same time remember information from a secondary task. The amount of information stored from the secondary storage task is thought to reflect working memory capacity. Use of this methodology is based on the assumption that more efficient processing of the primary task will be reflected in more resources being available to accomplish the secondary task. For example, Case, Kurland, and Goldberg (1982) used a counting span task where participants were presented with a set of white cards, one at a time. On each card there was a number of colored dots, which the participants were asked to count (primary task). After the last card was counted and removed, the participants were asked to recall the number of dots on each card (secondary task). The number of cards in each set was gradually increased, so the memory load became greater. The number of final counts recalled was considered an index of working memory capacity.

A distinction has been made between domain-specific and global working memory (Daneman & Carpenter, 1983; LeFevre, 1993). For example the problem 123 + 59 requires component processes such as calculation, carrying, and storage of intermediate solutions. As each task component (e.g., calculation) becomes more efficient and requires less of the limited working memory resources, more resources become available for other aspects of processing and storage (Ashcraft et al., 1992; LeFevre, 1993). In this view, a working memory task that taps the efficiency of task-specific processes will be the best measure of how important working memory is within a specific domain. To examine specific processes, researchers have developed complex memory span tasks. In these complex span tasks, a primary and secondary task are used but by manipulating the task specificity of the primary task, a distinction can be made between global and task specific processes.

Developmental Changes in Multiplication Skill and Knowledge

Theorists and researchers have agreed that to understand how knowledge is represented and manipulated in memory, it is important to understand how it is acquired and constructed. An effective means for studying the acquisition of knowledge is to examine the changes that occur with development. I examined age-related changes in (a) the computational skills used to solve simple multiplication problems including the procedures used and indices thought to reflect the representation of facts in memory, (b) concepts important for understanding multiplication, and (c) working memory important for mathematics.

Computational Skill

Different types of models provide different accounts of children's computational skills. For example, in ASCM, relations between memory for number facts and use of procedures are emphasized. In structural models, memory is organized via numerical indices (i.e., sum, sum square, and product) in fairly direct ways. No specific account of the selection and use of procedures is presented in problem-answer association models because the focus is on retrieval. Using these existing models as a reference, understanding age-related changes in computational skill entails studying at least two areas: (a) changes in the procedures children use to solve simple multiplication facts, and (b) changes in the representation of facts in memory.

First, children use multiple procedures to solve simple multiplication problems (Cooney et al., 1988; Siegler, 1988a; Lemaire & Siegler, 1995). What is less clear is how children acquire new procedures, and how changes in the frequency with which children use different procedures take place. A goal of my research is to use ASCM to explore changes in the use of procedures that occur during a time when children are learning multiplication facts.

A second goal of my research is to use structural accounts and ASCM to explore age-related changes in the representation of multiplication facts in memory[1]. Inferences regarding the representation of multiplication facts in memory are typically made using accuracy data and solution latencies. Hence, studying changes in memory representation really involves studying age-related changes in (a) accuracy and solution latencies, and (b) the effectiveness of hypothetical indices used to interpret accuracy and solution latency data. The most frequent empirical finding related to accuracy and latency for solving simple multiplication problems is the problem-size effect (Ashcraft & Christy, 1995; Campbell & Graham, 1985; Cooney et al., 1988; Koshmider & Ashcraft, 1991). Typically, the problem-size effect has been assumed to apply to all problems, regardless of how the problem was solved. Because structural variables (e.g., sum or product) account for the problem-size effect reasonably well, they have been used extensively to interpret accuracy and latency data in terms of the underlying structure of memory. Few researchers have evaluated whether other variables are useful for predicting accuracy and latency data, and whether age-related changes are evident in the effectiveness of different variables.

To examine age-related changes in the use of procedures and the representation of multiplication facts, children in Grades 4 and 6 were asked to solve 28 multiplication problems. Accuracy and latency were recorded, as were immediately retrospective self-reports on how the problem was solved.

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[1] Learning/experiential models are not discussed further because they share similar assumptions with ASCM regarding the influence of problem frequency on the representation of number facts, but do not include an account of the procedures children use.

Use of Procedures

Researchers have used overt behavior and self-reports to examine the use of multiple procedures in simple multiplication by children in Grades 2 to 3 (Lemaire & Siegler, 1995; Siegler, 1988a) and adults (LeFevre et al., 1996a). Very little research of this type has been conducted with children in later elementary grades (but see Cooney et al., 1988), despite the fact that multiplication facts are heavily practiced when children are in Grades 4 through 6.

Siegler and his colleagues (Lemaire & Siegler, 1995; Siegler, 1988a) used the overt behavior of children in Grades 2 and 3 when solving simple multiplication problems, to identify a number of procedures including retrieval, repeated addition, writing the problem, and counting sets. Use of self-reports, especially with older children and adults, often provides critical information not obtained from observed behavior. Using Grade 3 and 4 children's self-reports, Cooney et al. (1988) identified procedures similar to those Siegler discussed. They also identified other procedures, including use of derived facts, rules, and explanations. LeFevre and her colleagues (LeFevre et al., 1996a; LeFevre, Sadesky, & Bisanz, 1996b) evaluated adults' self-reports of how they solved simple arithmetic problems and found that adults use multiple procedures, not just retrieval, to solve both addition and multiplication problems.

Given that children in Grades 2 and 3 and adults have been found to use many procedures to solve simple multiplication problems, it is reasonable to expect that children in Grades 4 and 6 will also report using multiple procedures. At least two implications would arise from such a finding. First, models that account for multiple procedure use, such as ASCM, could be considered more helpful for understanding children's multiplication than models focused solely on retrieval from memory. Second, the finding that children use multiple procedures, even during a period of schooling when memorization and retrieval of multiplication facts is emphasized, would influence how development of multiplication is understood. Specifically, development is often described in terms of a progression through stages where a single procedure or way of thinking predominates. As children move from one stage to the next, more advanced procedures are thought to replace less advanced ones. If it is found that children at different ages use multiple procedures, then developmental accounts of multiplication must also emphasize variability in behavior. Any results obtained must be interpreted in terms of the constraints of using self-reports to measure behavior.

Self-reports of procedure use have been criticized on the basis that they may be misleading in two ways (Cooney & Ladd, 1992; Russo, Johnson, & Stephens, 1989). A self-report is nonveridical if it does not accurately reflect the underlying solution process. Errors of omission (e.g., not reporting some thoughts) and errors of commission (e.g., reporting mental events that did not occur) reduce the veridicality of self-reports. Another source of process data (i.e., accuracy or latency data) is necessary to evaluate the accuracy of self-reports (Russo et al., 1989). To address the issue of veridicality, latency and error data for each reported procedure were compared. If self-reports are veridical, for example, then reported use of retrieval should be associated with high accuracy and short latency when compared to backup procedures. Performance is reactive if self-report verbalization changes the primary process underlying performance. Tests of reactivity usually compare a silent control to the self-report condition. Although a silent control was not included in this study, data were compared to previous research on multiplication where self-reports were not used. More specifically, observed patterns of errors and latencies were compared with data from previous research. If self-reports are not reactive, the present results should be similar to previous research.

Changes in Use of Procedures

The development of multiplication skill has been characterized as a progression from the use of effortful procedures to retrieval from an associative network (Cooney et al., 1988; Geary, Brown, & Samaranayake, 1991). A more precise depiction of this progression must involve evaluation of the cognitive changes that occur. In a longitudinal study with Grade 2 children, Lemaire and Siegler (1995) found that children's behavior changed in ways consistent with predictions made in ASCM. Children used retrieval more frequently over time and the speed and accuracy with which each procedure was executed increased substantially. To determine whether the changes described in ASCM are useful for understanding the performance of older children, differences between the performance of children in Grades 4 and 6 were evaluated. Changes in the frequency with which each procedure was used were evaluated to determine whether retrieval becomes more dominant. Changes in the speed and accuracy with which procedures were examined to determine if use of specific procedures becomes more efficient. Confirming the hypotheses made in ASCM with older children is significant because the predictions in ASCM can provide the basis for generating greater insight into the development of multiplication procedures, beyond the simplistic description that effortful procedures give way to efficient ones.

Problem-Size Effect and Problem Characteristics

To study children's changing representation of multiplication facts, variables that index the problem-size effect and constraints that may qualify the interpretation of the problem-size effect were examined. Problem size, as indexed by product, accounts for approximately 40 to 83 percent of the variance in solution latencies for children (Campbell & Graham, 1985; Cooney et. al., 1988; Siegler, 1988a), and 36 to 42 percent for adults (Campbell & Graham, 1985; LeFevre et al., 1996a). Although problem size is a reliable predictor of latencies and errors, the structural variables used to index problem size provide an incomplete account of the variability in solution latencies on single-digit multiplication problems (LeFevre et al., 1996a). Problem characteristics can influence the effectiveness of product as a predictor of latencies. For example, ties (e.g., 3 x 3) and problems with 5 as an operand are solved more quickly than other problems of comparable magnitude (Campbell & Graham, 1985). Campbell and Graham reported that only problems with 4, 8, and 9 in them occupied the rank order position predicted by structural variables.

To determine the role of problem-size and problem characteristics in children's representation of multiplication facts, the effectiveness of these variables in accounting for variance in children's solution latencies was compared. If problem-size is an important determinant of performance in multiplication, then an empirical relation should exist between it and both accuracy and latency data; individual problems should not deviate markedly and systematically from this pattern. Further, problems with different operands but the same product should have similar accuracy rates and solution latencies. If, in contrast, specific problem characteristics are important in influencing Grade 4 and 6 children's performance, then simple multiplication problems should not all conform to patterns of accuracy and latency data predicted by the problem size effect.

If different patterns of accuracy rates and solution latencies are observed for different identifiable problem types, then it is plausible that children use different procedures selectively, depending on problem characteristics. In ASCM, Siegler and Shipley (1995) propose that problem characteristics are useful for predicting latency and accuracy. Specifically, knowledge is stored about the speed and accuracy of particular procedures on problems with specific features. If a procedure is fast and accurate on certain types of problems, then the probability that it will be used increases. For example, problems with operands of 5 are more often solved with a counting-string procedure (e.g., "5, 10, 15, 20, . . .") than with derived facts or repeated addition. Counting-string procedures are acquired earlier and may be less error prone than derived-fact or repeated-addition solutions (Ashcraft et al., 1984). According to ASCM, solving 5-operand problems with a counting-string procedure would be more likely to be accurate than using other backup procedures, resulting in higher levels of associative strength. To evaluate the role of problem characteristics in multiplication performance, the strategic use of procedures on specific types of problems was examined.

Problem-Size and Use of Procedures

If the use of different procedures plays an important role in children's multiplication performance, then the relations between solution latencies, the problem-size effect, and multiple procedure use must be examined. Again, few researchers have addressed the influence of individual procedures when accounting for solution latencies, or how the use of different procedures may mediate the problem-size effect. Backup procedures are related to longer solution latencies compared to retrieval (Siegler, 1988a). LeFevre and her colleagues (1996b) found that the problem-size effect in simple addition can be attributed largely to the use of only retrieval on smaller problems, and the use of backup procedures and retrieval on larger problems. Furthermore, they found the problem-size effect was greatly diminished on problems where retrieval was used, suggesting the associative links among problems are not structured entirely according to size. If the problem-size effect can be accounted for by the strategic use of procedures, then models that reflect the connections between multiple procedure use and memory may be more appropriate than structural models for understanding children's multiplication.

To examine the influence procedure use may have on solution latencies, two sets of analysis were conducted. First, for each grade, product was used to predict solution latencies on problems where retrieval and backup procedures were used, versus problems where only retrieval was used. If multiplication facts are stored via structural indices in memory, then the procedure used should not influence solution latencies. Consequently, product should be equally effective in predicting solution latencies for both sets of problems. If children do use procedures strategically on different problems, then the effectiveness of product in predicting solution latencies should vary depending on the procedures used. Specifically, if retrieval is used mainly on small problems, and retrieval and backup procedures are used on large problems, then product should effectively predict solution latencies when all trials are considered. However, the effectiveness of product should decrease when only retrieval trials are considered. Differences between Grades 4 and 6 in the effectiveness of product in predicting solution latencies depending on the procedures used should be important for understanding the development of the representation of math facts.

In the second set of analyses, variables derived from structural models and ASCM were compared directly. Specifically, product and percent retrieval were used to predict children's solution latencies. According to ASCM, probability of using retrieval reflects the distribution of associations between problems and answers, and so probability of retrieval was used as the primary index of associative strength. If memory/procedural accounts such as ASCM are more appropriate than structural accounts, then the associative index should better predict solution latencies. Further changes in the relations between structural or associative variables were examined between Grades 4 and 6 in order to evaluate the influence of different variables on the developing representation.

Conceptual Knowledge

The acquisition of multiplication skill and knowledge involves more than memory for facts and the use of procedures. Conceptual knowledge has been demonstrated to be important for the development of mathematical skills (Baroody, 1995; Geary et al., 1992). Consequently, a detailed examination of conceptual development seems essential for studying the acquisition of multiplication skill and knowledge. Two questions related to conceptual understanding and the development of multiplication were addressed: (a) Does conceptual knowledge for multiplication develop with age and, if so, what is the course of that development? and (b) Do measures of the product of children's thinking (i.e., number correct) reflect conceptual knowledge? Grade 4 and 6 children solved a number of problems that involved either the direct application of concept or the use of a concept to enable a novel operation. The problems were based on a number of concepts and domains important for multiplication, including: commutativity, repeated addition, number sense, part-whole relations, word problems and proofs.

Development of Conceptual Knowledge

If children's conceptual knowledge develops during a period of schooling when so much emphasis is put on memorization and computation, then Grade 6 children should understand more concepts than Grade 4 children. Further, the distinction between direct and enabling applications has implications for understanding the development of concepts related to multiplication. Often conceptual development is thought to occur in an all-or-none fashion, and many theories of cognitive development have involved simple descriptions of these all-or-none changes in children. In Piaget's theory, stages of development are defined by the specific conceptual advances that occur within a particular period of time. A child in the concrete operational stage understands reversibility and a child in the pre-operational stage does not. However, conceptual development may involve the uneven advancement of concepts, with a child being in different "stages" depending upon the specific concept being evaluated. In fact, the uneven development of concepts has been observed in children's understanding of the use of metaphors in different domains (Kiel, 1989). Demonstrating that conceptual development in multiplication occurs unevenly, instead of in an all-or-none fashion, changes the focus of research from describing absolute changes in children's thinking to that of trying to understand the mechanisms through which concepts change gradually.

By studying the many concepts important for multiplication and how they are applied, we can evaluate whether knowledge develops in an all-or-none or uneven manner in multiplication. Specifically, it is reasonable to expect that the direct application of concepts in multiplication would be evident in children's reasoning earlier than enabling concepts because some direct-application of a concept is required before an enabling function can be demonstrated. If the development of conceptual knowledge in multiplication is uneven, then understanding of direct-application concepts should remain relatively constant between Grades 4 and 6, but understanding of enabling application concepts should improve.

Product Measures and Conceptual Knowledge

Assessment of children's performance in multiplication most often involves examining the number of problems solved correctly on an achievement test. Number correct on an achievement test may reflect the product of children's thinking, but it is questionable whether it reflects the processes children use. Understanding the processes children use to solve problems is important for theory and remediation. Does number correct reflect children's understanding of multiplication as reflected in their knowledge of concepts important for multiplication? If number correct accurately reflects children's performance, it should be closely related to measures of conceptual knowledge.

Working Memory

Researchers have found that working memory constrains the performance of adults on simple counting and single digit arithmetic problems (Ashcraft et al., 1992; Logie et al., 1994). If subjects continuously repeated letters or words, presumably suppressing sub-vocal rehearsal in the articulatory loop, then counting and single-digit addition performance was impaired (Ashcraft et al., 1992; Logie et al., 1994). How might sub-vocal rehearsal in the articulatory loop be involved in simple arithmetic? In the case of simple addition the initial problem may be stored in the articulatory loop while a procedure is used to calculate or retrieve the answer (Logie et al., 1994).

As well, researchers have examined the age-related changes in working memory. Siegel and Ryan (1988) found that, from the ages of 7 to 13 years, children's working memory improved for semantics, syntax in sentences, and simple counting spans. Case et al. (1982) observed an increase in children's counting span from ages 6 to 12. They argued that improved working memory capacity is related to an increase in the efficiency with which children process information. As less space is required for operating on information, more becomes available for storage.

Researchers have not examined age-related changes in working memory important for arithmetic. A clear account of the role of working memory in mathematics must included development. Simple counting spans may not reflect the demands placed on working memory when it is used in arithmetic. Consequently, age-related changes in working memory measures thought to involve mathematical processing were examined.

Working Memory Tasks

To evaluate development in working memory when numbers are being processed, two tasks were used: backward digit span from the Wechsler Intelligence Scale for Children, and an operation span. Often working memory tasks are used in research without a clear theoretical basis or a detailed analyses of what the task are intended to measure. Consequently the construct of working memory is often ambiguous. To avoid this ambiguity, a detailed analysis of the tasks used in this paper is presented. In the analysis the components of the working memory tasks are identified and what they are intended to measure is discussed.

Backward digit span. Backward digit span involves reversing the order of a series of numbers (primary task), and then recalling the numbers in the reversed order (secondary task). Because it does not involve processing arithmetic operations, it is considered to be a measure of global working memory capacity. According to the Baddeley model, the task involves processing by the central executive and storage by the central executive in the articulatory loop. Global processing may take the form of strategic chunking of the digit into meaningful units by the central executive. Storage may involve sub-vocal repetition of the digits in the articulatory loop. Because simple digit spans do not involve arithmetic operations, these tasks may not reflect the demands placed on working memory in arithmetical problem-solving tasks.

Operation span. The operation span task is a complex measure of working memory that more closely approximates the demands placed on children when they are solving multiplication problems. In this task, which is a modification of the complex span used by LeFevre (1993), children had to solve a series of multiplication problems (primary task) and at the end of the series recall the answers to each problem (secondary task). Researchers have assumed that complex span tasks reflect only domain specific processing (Daneman & Carpenter, 1983; LeFevre, 1993). We argue that complex span tasks involve both global and domain specific processing. The operation span task presumably involves the same global storage of numbers through the articulatory loop that the backward digit span does. It is influenced by domain-specific processing in that if an individual's math skills are inefficient, then more resources will be required for processing, and storage in articulatory loop will be limited. As well, operation span may also involve global working memory processing, such as, placing the answers in serial order or making connections between answers.

In the section on Individual Differences, the global and domain-specific components of operation span are examined by using backward digit span and simple measures of multiplication processing to predict individual differences in performance on it. The purpose now, however, is to examine age-related changes. If working memory related to numbers and simple arithmetic increase with age then children in Grade 6 should have longer memory spans than children in Grade 4 on the backward digit span and the operation span task.

Individual Differences and Mathematical Achievement

Studying age-related changes in groups of children has provided important information regarding the general course of development in mathematics. However, much variability exists in the performance of children. Individual children progress at different rates in the acquisition of mathematics skill and knowledge, and attain different levels of mathematical proficiency (Siegler, 1988b). The consideration of individual differences provides valuable information regarding constraints on skilled mathematical performance, different developmental outcomes, and the relations among measures important for mathematics performance.

When multiple measures are used to evaluate mathematical ability, as is the case presently, an important issue to address is what measure of mathematical ability should be used as the criterion on which individual differences are assessed. Achievement tests are used extensively in educational, clinical, and research settings to measure proficiency and development in mathematics. These tests are rarely evaluated in terms of relations to different areas of cognitive functioning important for mathematics. Consequently, individual differences in mathematical achievement for children in Grades 4 and 6 were evaluated using computational, conceptual, and working memory measures related to multiplication. Analysis of individual differences provides information on how achievement is constrained by children's computational skill, conceptual understanding, or working memory capacity. As well, different developmental outcomes were assessed by evaluating the changing relations from Grade 4 to Grade 6 in mathematical achievement and measures of multiplication performance.

If computational skill, conceptual knowledge, working memory, and mathematical achievement are examined in isolation, a fragmented picture of children's thinking is portrayed. Mathematical cognition, like all complex human behavior, involves the integration of different cognitive processes. Consequently, individual differences were also used to identify the overall relations between mathematical achievement, computational skill, conceptual understanding, and working memory. Specifically, mathematical achievement was no longer used as criterion of mathematical performance. Instead, cluster analysis was employed to evaluate the relations among all the different areas of cognition important for multiplication.

Computational Skill

Individual differences are important for understanding arithmetic performance in children and adults (LeFevre et al., 1996a; Lemaire & Siegler, 1995; Siegler, 1988b). LeFevre et al. (1996a) found that adults varied widely in the extent to which they rely on direct retrieval, and individual differences in latencies and patterns of performances were linked to the degree to which participants routinely uses direct retrieval on simple multiplication problems. Siegler (1988b) found consistent individual differences in first graders' choice of procedures in addition, subtraction, and reading. Good children, not-so-good children, and perfectionists differed in their knowledge about problems, the stringency of thresholds for stating retrieved answers, and on an achievement test. Perfectionists were children who set very high confidence criteria and had peaked distributions. Good children also had peaked distributions but set less high confidence criteria. Both perfectionists and good children did well on an achievement test. Not-so-good children set lower confidence criteria, had less peaked distributions, and did poorly on the achievement test compared to the other children. In a longitudinal study, Lemaire and Siegler (1995) demonstrated that individual differences in the early execution of backup procedures predicted efficiency in later use of retrieval. They argued that more accurate early execution of backup procedures leads to stronger associations between each problem and its correct answer, and to weaker associations between the problem and incorrect answers. Consequently, correct answers are retrieved more frequently.

To examine the role of computational skills in relation to individual differences in achievement, correlations between standard achievement and measures of computational skill in multiplication were examined for each grade. Given that (a) individual differences exist in children's and adults' use of procedures and representation of multiplication facts, (b) these individual differences are related to proficiency in multiplication, and (c) the emphasis on calculation of facts in most measures of mathematical achievement, then it is expected that accuracy, solution latency, and procedure use for solving simple multiplication problems should be correlated with mathematical achievement. As well, examining differences between children in Grades 4 and 6 in the relations between computational skill and mathematical achievement should be informative regarding different developmental outcomes. Specifically, comparing the pattern of relations for Grade 4 students to those of students in Grade 6 should be suggestive regarding possible developmental courses that would led to the observed changes.

Conceptual Knowledge

Unfortunately very little research exists on individual differences in conceptual understanding. The distinction between direct and enabling applications has implications for understanding individual differences in children understanding of multiplication. Specifically, individual differences may exist in children understanding of direct versus enabling applications. Consequently, I addressed whether individual differences exist in children's understanding of concepts important for multiplication. Researchers have argued that understanding mathematical concepts is important for the early development of computational skills (Baroody, 1995; Geary et al., 1992). If conceptual understanding is important for the initial development of computational skills, then mathematical achievement and measures of conceptual knowledge should be related, at least for children in Grade 4. Presumably, by Grade 6, children have had a substantial amount of practice and have learned computational skills. Hence, measures other than conceptual understanding may be more important for predicting mathematical achievement.

Working Memory

Because of its limited capacity, variability in working memory has been thought to be an important source of individual differences in complex tasks such as reading and mathematics (Ashcraft et al., 1992; Daneman & Carpenter, 1980; LeFevre, 1993). Researchers have examined the development of working memory in children (Case et al., 1982; Siegel & Ryan, 1989) and they have evaluated the relations between working memory and arithmetic in adults (Ashcraft et al., 1992; Hitch, 1978; Logie et al., 1994). Researchers have not evaluated age-related changes in the relations between working memory and arithmetic. The changing relations between working memory and arithmetic were evaluated in two steps. First, the underlying components of the operation span task were probed to ensure a clear understanding of working memory. Second, individual differences in the relations between working memory and arithmetic were explored and age-related changes in these relations were examined.

Global Versus Domain-Specific Working Memory

The distinction between global versus domain-specific working memory is important for understanding individual differences. Individuals may vary in the total amount of resources they have available for processing in the central executive (global working memory), and may also vary in the efficiency with which specific processes use central executive resources (domain specific skills) (LeFevre, 1993). When using a complex span task, by manipulating the specificity of the primary task a distinction can be made between individuals who may have the same total capacity but vary in the efficiency with which they execute task-specific process (LeFevre, 1993). Presumably, a working memory task that taps the efficiency of task-specific processes may be the most potent predictor of skilled performance. Based on this assumption, the operation span may be the best predictor of skilled arithmetic performance. However, before this span task is used to predict performance it is important to evaluate whether it does indeed reflect global and task-specific processes.

If operation span measures global working memory and arithmetic-specific processes, then individual differences in both backward digit span and simple measures of multiplication skill should be correlated with individual differences on operation span. Backward digit span should correlate with individual differences on the operation span because both presumably involve the same storage of numbers through the articulatory loop, and global working memory processes. If operation span also involves domain-specific processing, simple measures of mathematics processing should predict individual differences on the operation span. Hence, the proportion of times children use retrieval when solving simple multiplication problems should influence the operation span. Children who use effortful, repeated-addition procedures more frequently than retrieval should have fewer resources to hold the answers in the articulatory loop, and not remember as many answers. Median solution latency for simple multiplication problems should reflect the operational efficiency of children's processing (Case et al., 1982). Children who recall the answers on the operation span very quickly would not have to hold the series of answers in memory for a long time, reducing the possibility of decay. To evaluate whether the operation span reflects both global and domain-specific working memory processing, I used children's backward digit span, proportion retrieval on simple multiplication, and median latency on simple multiplication to predict individual differences in operation span.

Working Memory and Mathematics Achievement

Often the goal of using a working memory measure is to account for individual differences in skilled performance on a complex task such as reading or arithmetic. For example, researchers have found that adults' skilled performance on a variety of reading measures is related to a large reading span (Daneman & Carpenter, 1983: LeFevre, 1993; but see Turner & Engle, 1989). As well, skilled performance in arithmetic is associated with larger mathematical operation spans (LeFevre, 1993). I examined whether working memory capacity in children predicts individual differences in arithmetic achievement. If working memory, either general or specific, is related to mathematical skill in children, then measures of working memory should correlate with measures of arithmetic, such as mathematics achievement and problem-solving in multiplication. As well, the extent to which measures of mathematics-specific process in working memory and measures of global resources in working memory predicted individual differences in mathematics achievement was evaluated.

A number of tasks important for measuring the different components of working memory have been described in this paper. Further, it has been argued that (a) backward digit span reflects global processing and storage, (b) operation span involves global processing, domain-specific processing, and storage, and (c) simple measures of arithmetic, such as mean solution latency for multiplication problems and percent use of retrieval on multiplication problems, reflect domain-specific processing efficiency. To understand the changing relations between working memory and arithmetic between Grades 4 and 6, it is necessary to evaluate the relative importance of these different measures of working memory at each grade. First, the relative importance of the two span tasks for predicting individual differences in mathematics achievement in each grade was evaluated. If domain-specific processes are more important, then operation span should be a better predictor than backward digit span. If global processes are more important, then backward digit span should be a better predictor than operation span.

Previous researchers have only included the span tasks in their analyses of the importance of working memory for accounting for individual differences in domains. They have assumed that the domain-specific operations required in their complex span tasks are a sufficient measure of processing efficiency. I included specific measures of processing efficiency, along with the memory span task, to further clarify the role of domain-specific processes. Specifically, in a second set of analyses the span task that was the best predictor of mathematics achievement based on the previous analyses was used, along with a measure of processing efficiency, to predict individual differences in mathematics achievement. Using this approach, the combined influence of different components of working memory on arithmetic could be examined.

Relations Among Measures

Researchers have found that conceptual knowledge is important for the development of later computational skills in addition (Baroody, 1995; Geary et al., 1992). Geary et al. (1992) found that counting procedures in young children with mathematical disabilities are related to their understanding of essential and unessential concepts of counting (i.e., one-to-one correspondence). If conceptual knowledge is important for the early development of multiplication procedures, then the relations between conceptual knowledge and computational skill should be greater for Grade 4 than for Grade 6 children.

Working memory may also be related to children's procedure use. For example, when using a counting procedure to solve an addition problem, if the original representation of the problem's integers decays quickly in working memory, then the answer generated by this count and the original representation of the problem would not become associated in long-term memory. As a result, the probability of using retrieval to solve the problem in the future would not increase. Further, efficient use of mathematical procedures may leave more resources available for storage in the working memory system. Because of the close association between procedure use and memory, we would accept that working memory and computational skill should be highly related for both Grade 4 and Grade 6 children.

The relations between computational skills, conceptual knowledge, working memory, and mathematical achievement were examined in two ways. First, for each grade the correlations among multiple measures of children's multiplication skill were calculated, including: (a) the proportion of retrieval use on simple multiplication problems, (b) solution latency for simple multiplication facts, (c) raw score on the WRAT-III, (d) proportion of principally based responses for commutativity/repeated addition, related-fact problems, greater than/equal-to, and estimation problems, (e) proportion of principally based responses for word problem, (f) backward digit span, and (g) operation span. Correlations only provide information about how closely related each measure is to the next one. Consequently, hierarchical cluster analysis was also used to detect the interrelations among the set of measures separately for Grades 4 and 6. In hierarchical cluster analysis each measure is considered as a separate cluster, and then the two most similar variables are joined to form a cluster. Similarity is determined through association or distance between variables. The amalgamating process continues in a step-wise fashion (joining variables or clusters of variables) until a single cluster is formed that contains all the variables. Relations among variables can be established by evaluating which variables cluster together first, and the order with which variables cluster.

METHOD

Participants

Sixty children from Grade 4 (30 females, 30 males) and 60 children from Grade 6 (30 females, 30 males) were assessed individually in two sessions. Grade 4 children ranged in age (in years:months) from 9:2 to 11:3 years, with a median age of 9:11 years. Grade 6 children ranged in age from 11:3 to 13:2 years, with a median age of 11:11 years. Studying the performance of children in Grades 4 and 6 allows for the examination of a range of experience in multiplication. According to the curriculum guidelines for the province of Alberta Education, multiplication is taught for the first time in Grade 4 and hence it is a relatively novel domain for children. By Grade 6, children have had substantial instruction and practice in many areas of multiplication.

Procedures and Materials

Children were assessed individually in two sessions lasting approximately 30 minutes each. In the first session an arithmetic achievement test, a multiplication computation task, and tests of working memory were administered. In the second session tasks evaluating conceptual knowledge were administered.

Standardized Achievement Test

The Wide Range Achievement Test, third revision (WRAT-III; Jastak & Jastak, 1993) arithmetic subtest was used to measure general computational ability. The WRAT-III has been used extensively in practice and research to assess children' achievement and to identify children with learning disabilities (Goldman, Pellegrino, & Mertz, 1988; Rourke & Strang, 1983; Siegel & Ryan, 1989; Strang & Rourke, 1985). The arithmetic subtest consists of addition, subtraction, multiplication, and division problems as well as problems on measurement, fractions, percentages, and decimals. Children were given 10 minutes to answer as many problems as they could. Specific instructions for the subtest are found in Appendix A. Raw scores were converted to standard scores using the age-based norms provided in the WRAT-III.

Computational Task

The computational task was used to measure children' performance when solving simple multiplication problems. The problem set consisted of 28 combinations of multiplicand (2-9) and multiplier (2-9), excluding tie problems (e.g., 2 x 2). Only one of each commuted pair of problems (e.g., 3 x 7 but not 7 x 3) was presented. A second order was created by reversing the first. The complete set of problems is listed in Appendix B.

Problems were presented on a computer monitor controlled by a microcomputer. First the children were asked to solve each multiplication problem and answer aloud as quickly as possible without making any mistakes. Specific instructions for the task are found in Appendix A. Problem presentation and timing were controlled by a microcomputer connected to a microphone through a voice-activated relay. Children initiated each trial by saying "go" after a fixation point (an asterisk) appeared on the computer screen. Following a 100-ms tone and a 700-ms blank interval, the problem appeared and timing was initiated. When presented at a distance of 0.6 m, the problem subtended a visual angle of 3.3° vertically and 1.15° horizontally. The children's vocal response activated a voice key. Each distinct latency was counted and recorded in order to decrease the number of spoiled trials. In the few cases where the children made more than one response (e.g., coughed before responding), the experimenter keyed in the count for the intended response and the corresponding latency was recorded. The experimenter recorded the children's answer on the computer. Correct answers were followed by a high-pitched tone and incorrect answers were followed by a low-pitched buzzer. If no vocal response was detected within 30 seconds, the trial was terminated. Following the response, the cue "How Did You Get Your Answer?" appeared on the screen, prompting children to describe the procedure that they used.

Based on the children's overt behavior and verbal self-report, procedures were coded on-line by the experimenter according to previously defined categories. Responses were classified as retrieval when children claimed that they remember the answer, "just knew it," or solved the problem "from memory" and there was no evidence of overt calculations. Special trick procedures involved use of previously learned information to help solve a current problem. Two kinds of special tricks were coded. Derived-fact procedures involved the use of known arithmetic facts to derive solutions. For example, for the problem 6 x 7 a children might report "I know that 6 x 6=36, and so 6 more would be 42." Nines rules were algorithms particular to nine-times problems. For example, for 4 x 9, the children would hold ten fingers and identify the fourth finger. He or she would then count the number of the fingers before the fourth fingers and state this as the tens number (3), and count the number of fingers after the fourth finger and state this as the ones number (6). Repeated addition procedures involved adding an operand the appropriate number of times, for example, adding 5 + 5 to solve 5 x 2. Counting string procedures involved use of a memorized string to produce the answer, for example, solving 5 x 7 by counting "5, 10, 15, 20, 25, 30, 35". Responses were classified as guess if the children stated they guessed the answer. Other denoted ambiguous responses or responses that did not fit the other categories.

Conceptual Tasks

Conceptual understanding was assessed with a series of tasks relating to the underlying principles of multiplication, and to the ability to generalize previously learned knowledge to novel stimuli and situations. All of the problems included double-digit numbers so that they could not be easily solved using retrieval. The full text of instructions for all the conceptual tasks is found in Appendix C.

Comparison Task

For the Comparison task children were presented with a piece of paper with two adjacent columns of 18 pairs of arithmetic problems, each with an answer. Children were told that the answer for the left problem in each pair was correct. The task was to determine, based on the information provided on the left, whether the answer to the corresponding problem on the right was correct. The children were instructed to do so without multiplying. Problems on the left and right were related by three different principles: repeated addition (8 pairs); commutativity (6 pairs); and derived or related facts (4 pairs). A complete list of the stimuli for the comparison task is found in Appendix D. According to the Alberta Education Program of Studies (1994), an educational goal for Grade 4 children is that they be able to multiply whole numbers by one- and two-digit whole numbers. Consequently, all problems involved multiplying 2-digit and 1-digit numbers. For repeated addition, commutativity, and related-fact problems, the 2-digit numbers were between 10 and 90, and the 1-digit numbers were between 3 and 9.

For the eight repeated-addition pairs, the problem on the left side was a correct repeated-addition problem and the problem on the right side was a multiplication problem. Half of the multiplication problems on the right were equivalent to the repeated addition problem on the left (e.g., 28 + 28 + 28 + 28 = 112 paired with 28 x 4 = 112). For the rest of the pairs, the 1-digit multiplier for the problem on the right was not equal to the number of addends in the left problem, but the 2-digit multiplier and product were the same (e.g., 28 + 28 + 28 + 28 = 112 paired with 9 x 28 = 112).

For the six commutativity pairs, the problem on the left side was a correct multiplication problem and the problem on the right included the same multiplier and multiplicand but in the reverse order. For half of the pairs, the products for the left and right were the same (e.g., 8 x 59 = 472 paired with 59 x 8 = 472). For the rest of the pairs, the product on the right was greater or less than the product on the left (e.g., 4 x 64 = 256 paired with 64 x 4 = 236).

For the four related-fact pairs, the problem on the left side was a correct multiplication problem. The problem on the right side had the same multiplicand, the multiplier was increased or decreased by one, and the product was different than the problem on the left. For half the problems the answer on the right was the correct product (e.g., 35 x 4 = 140 paired with 34 x 4 = 136), and for half the problems it was an incorrect product (e.g., 37 x 5 = 185 paired with 38 x 5 = 205). Related-fact problems are similar to derived-fact problems and useful in evaluating children' understanding of part-whole relations and their ability to use a set of known number facts to derive the solutions to unknown combinations.

Six orders for the comparison task were created based on the following constraints: (a) half of the orders started with a true problem and half started with a false problem; (b) different problem types were evenly distributed across the list of 18 problem pairs in a mixed fashion; (c) true and false problems were evenly distributed and; (d) 2 orders started with a repeated-addition problem, 2 with a commutativity problem, and 2 with a related-fact problem.

Number Sense Task

Children were asked to determine, without multiplying or dividing, whether the answers to multiplication problems were probably right or probably wrong. Problems were designed so that three principles of number sense could be used by the children to judge whether the answer to a problem was correct: the relative effects of operations on numbers, the allowable minimum magnitude of a product in multiplying two whole numbers, and estimation. Number sense problems involved simple multiplication of a 2-digit multiplier between 10 and 90 and a 1-digit multiplier between 3 and 9.

A complete list of the stimuli for the number sense task is found in Appendix E. For the three operand-equal-to-product problems, the answer given was equal to one of the operands, and the other operand was not equal to 1 (e.g., 32 x 6 = 32). These problems were designed to assess children's ability to recognize that the product can not equal the multiplicand if the multiplier is a number other than one. For the three operand-greater-than-product problems, the answer given was less than the larger operand (e.g., 5 x 67 =

55). These problems were designed to assess use of a principle that specifies the allowable minimum magnitude of the product in multiplying two whole numbers (i. e., that the product must not be less than both operands). Three estimation problems were presented in which the answer given was close to the correct answer (e.g., 72 x 7 = 514). Children were to estimate to determine whether the answer was probably right or probably wrong. These problems were designed to assess children' ability to use their understanding of place value to generate an answer close the correct answer.

Six problem orders were created based on the following constraints: (a) two orders started with estimation problems, two started with product-less-than-multiplicand problems, and two started with product-equal-to-multiplicand problems; and (b) different problem types were evenly distributed through the task in a mixed fashion.

Word Problems Task

Four problems were designed to assess children' ability to apply the principles of multiplication in novel, non-routine tasks. Children were asked to report how they would solve each problem, but they did not need to carry out the computational steps and provide an answer. Stimuli for the word problems are found in Appendix F. For the repeated addition/irrelevant information problem, children were required to apply the principle of multiplication as repeated addition to determine the correct steps needed to solve the problem. They also needed to ignore irrelevant information in the problem. For the insufficient information problem children needed to realize that the problem does not provide enough information for it to be solved. The multiplicative compare problem involved three steps. Children needed to identify which numbers they needed to multiply, identify which numbers they needed to add, and realize that they needed to compare their answer with a number given in the problem. The Cartesian multiplication problem required children to combine elements of two sets orthogonally. Four problem orders were presented.

Proofs Task

For the Proofs task children were given manipulatives (beads), and asked to prove, using the manipulatives, why a certain product was the correct answer for a specified question. For example, children were asked to prove, using the beads, that 3 x 4 = 12. Three problems of this type were administered. The presentation of problems was varied to create 9 orders. A complete list of the stimuli for the proofs task is found in Appendix G.

Working Memory Tasks

Backward Digit Span

The backward digit span subtest from the Wechsler Intelligence Scale for Children (WISC-III) was used as an index of general working memory capacity. Participants were presented with a series of numbers and asked to recall the numbers in reverse order. Digits were presented orally in series progressing from 2 to 8 digits. Two series of the same length were presented before incrementing the number of digits in the series by one. Consequently 7 pairs of series were presented, for a total of 14 number series. The backward digit span stimuli are found in Appendix H. Children received one point if they accurately recalled all the numbers in the correct order for one of the number series a group. If children failed to accurately recall all the numbers in the correct order for both series of a particular length, the task was discontinued. Children could score a maximum of 14 points on the backward digit span task.

Operation Span

The operation span task was used to evaluate the role of mathematics processes in working memory. As a primary task, children were asked to produce the answer to a series of multiplication problems. Simultaneously, the children were required to remember the answer they gave for each problem in the series. As a secondary, task children were asked to recall the answers for each of the problems after viewing all the problems in a series. Problems were presented visually in series of lengths 2 through 5, one problem at a time. Four lengths of problems were presented, with two problem series in each length for a total of 8 problem series. Problems presented in the operation span were identical to the problems presented in the multiplication task to allow for analysis of the relations between simple multiplication processes and working memory ability. A complete list of the stimuli is found in Appendix I. Children received one point for accurately recalling all the numbers they had given as answers for a problem series. If children' failed to accurately recall all the numbers they had given as answers for both problem series of a particular length, they received no points and the task was discontinued. The maximum score that could be obtained on the operation task was 8 points.

RESULTS AND DISCUSSION

Developmental Changes in Multiplication Skill and Knowledge

Computational Skill

Results are reported in five sections. First, analyses of error and latency data are presented to indicate the difficulty of the task and to evaluate whether the use of self-reports influenced children' performance. Second, children's use of procedures is examined, including analyses of the veridicality of self-reports. The third section focuses on changes in the use of procedures between Grades 4 and 6. Fourth, the value of using structural variables to account for the problem-size effect is examined. Finally, regression analyses are used to compare associative and structural variables in terms of their ability to account for the problem-size effect.

Errors and Latencies

Errors. For Grades 4 and 6 respectively, 1.6% and 1.8% of trials were invalid due to malfunctioning of the voice-activated relay. These trials were excluded from all analyses. As well, children in Grades 4 and 6 were unable to produce an answer within 30 seconds on 1.3% and .06% of valid trials, respectively. These trials were coded as errors and response latencies were estimated at 30 seconds. Grade 4 children were incorrect on 9.0% of valid trials, a rate lower than previously reported for children in Grades 3, 4, and 5. Grade 6 children were incorrect on 1.6% of valid trials, a rate lower than previously reported for adults (see Table 1).

Children' errors were categorized into a number of different types. An operand-related error is an incorrect answer that is the product for another problem that shares an operand with the present problem (e.g., 8 x 3 = 21). An operand-unrelated error is an incorrect answer that is the product for another problem that does not share an operand with the present problem (e.g., 8 x 3 = 28). A non-table error is a responses that is not a correct product to any single-digit multiplication problem (8 x 3 = 23). Following Siegler (1988a), close-miss errors also were identified. Close-miss errors were responses that were within 10% of the correct product for the 26 problems with products of 10 or more. For problems with products smaller than 10, no incorrect answer could be within 10% of the product. All close-miss errors were coded as either operand-unrelated or non-table errors.

The percentage of each type of error, excluding close-miss errors, is shown in Table 2, with the values from Campbell and Graham (1985) included for comparison. We compared our data with Campbell and Graham's because they had participants of similar ages and detailed analyses of errors. Consistent with Campbell and Graham's research, operand-related errors were the most frequent type for children in Grades 4 and 6. The distributions of errors across types in Grades 4 and 6 were similar to patterns exhibited by Campbell and Graham's children in Grades 3-5. Changes in error patterns were also similar to changes observed by Campbell and Graham: with increasing age the proportion of operand-related errors increased and the number of operand-unrelated and non-table errors decreased. When eligible operand-unrelated and non-table errors were identified as close-miss errors, Grade 4 children displayed a similar number of errors as Grade 3 children in previous research (24% vs. 29%) (Siegler, 1988a). There are no comparable subjects in previous research for children in Grade 6 on this measure.

Latencies. Mean latency was calculated by identifying the median latency for correct problems for each child and averaging across children for each grade (see Table 3). Grade 4 children's mean latency (2.56 s) was marginally faster than Campbell and Graham's (1985) children in Grades 3 and 4 and considerably faster than the means obtained by Siegler and his colleagues for children in Grades 2 and 3 (Siegler, 1988a; Lemaire & Siegler, 1995). Grade 6 children's mean latency (1.53 s) was somewhat faster than Campbell and Graham's Grade 5 children but slower than their adults. Implications of our analyses of error and latency data for evaluating use of self-reports are discussed in the Discussion.

Use of Procedures

We examined the frequency and distribution of procedures used to solve the 28 multiplication problems. The percentage of valid problems on which each procedure was used, the percentage of children using each procedure at least once, the mean across children of median latencies for correct trials for children using the procedure at least two times, and the accuracy for each procedure are shown in Table 4.

Procedures. Children in both grades used retrieval and three backup strategies: special trick, repeated addition, and counting string. Most children in Grade 4 and all children in Grade 6 reported using retrieval at least once (see Table 4). For both grades retrieval was used most frequently across trials, followed by special trick, repeated addition, and counting string, respectively. Diverse use of procedures was evident for individuals as well. Of the children in Grade 4, 87% used two or more procedures, 63% used three or more, and 32% used at least four. For Grade 6, 58% used two or more procedures, 22% used three or more, and 3% used at least four. Thus younger children tended to use more procedures than older children, but even in Grade 6 most children used multiple procedures to solve simple multiplication problems.

In both grades the various nonretrieval procedures (i.e., repeated addition, counting string, and special trick) were slower than retrieval; use of a special trick was particularly slow (see Table 4). In Grade 4, children were almost 100% accurate when they reported using retrieval and were less accurate when they used nonretrieval procedures.

Changes in Use of Procedures

Increased use of retrieval. Although all children used multiple procedures, changes between Grades 4 and 6 occurred in both the mean number of procedures used and the proportion of times each procedure was used. Mean number of procedures used decreased from 2.86 for Grade 4 to 1.85 for Grade 6 children, $F(1, 118) = 29.84$, $p < .05$. Children in Grade 6 reported using retrieval more frequently than children in Grade 4, $F(1, 118) = 19.68$, $p < .01$, and using special trick, repeated addition, and counting string less frequently than children in Grade 4, $Fs(1, 118) < 6.93$, $ps < .01$ (see Table 4).

Increased efficiency of procedures. Changes occurred in how efficiently Grade 4 and Grade 6 children used procedures. For each procedure I calculated the percent correct and median latency for children using the procedure at least twice across trials. Use of retrieval was more accurate and faster in Grade 6 than in Grade 4, $Fs(1,119) > 9.66$, $p < .01$ (see Table 4). This improvement is even more striking given the fact that Grade 6 children used retrieval on a broader set of problems than Grade 4 children. Effective use of backup procedures also increased substantially between children in Grade 4 and 6 in terms of accuracy and speed, $Fs\ (1, 79) > 3.96$, $ps < .05$.

Problem Size

For each grade we calculated a median latency across children for each of the 28 problems. Only correct trials were used to calculate these latencies. Median latency was then correlated with the product of the operands. Product accounted for a significant amount of variance in solution latencies in Grades 4 ($r^2 = .43$) and 6 ($r^2 = .47$).

Problem characteristics. To evaluate whether product predicts solution latencies effectively for problems with different characteristics, we plotted mean latency as a function of product. We then inserted onto the plot the regression line for product as a predictor of latency. Finally we labelled each of the plotted points with the specific problem they represented (see Figure 1). Many exceptions to the problem-size effect were evident. For example, for both grades many problems with the same or similar product had very different latencies. Both 2 x 6 and 3 x 4 have the product 12, but the mean latency for 3 x 4 was much greater than it was for 2 x 6. Similar discrepancies were evident for 2 x 9 compared to 3 x 6 and 5 x 7 compared to 8 x 4. There were also systematic differences between the problems that fell above and below the regression line. For Grade 4, problems including 2 and 5 all fell below the regression line, as did 9 x 8.

The same distribution was evident for Grade 6 with two exceptions (7 x 3 and 2 x 8). Although product is a moderately good predictor of latency, examination of specific problems revealed that problem characteristics also influence latency. Specifically, problems including 2s and 5s are solved more quickly than would be expected based on product-size alone.

To identify the contribution that specific problem characteristics may make in accounting for latencies, two regression analyses were carried out. First, latency was regressed on product, and then a variable that coded whether a problem had a 2 or a 5 in it was added to determine whether it made an independent contribution (Model 1). Subsequently, the order of entry of the variables was reversed (Model 2). As shown in Table 5, both product and the 2s and 5s variable, entered first, explained significant variance in the latencies for children in Grades 4 and 6. Importantly, the unique contribution of 2s and 5s taken together was larger than that of product for Grade 4 (.31 versus .08) and Grade 6 children (.22 versus .14). These results support the hypothesis that problem characteristics are at least as effective as product in predicting latencies.

Based on our analyses, the use of product to demonstrate the appropriateness of a structural model of multiplication may not be warranted. The fact that different patterns of latencies are observed for certain types of problems raises the possibility that children may use different procedures selectively, depending on problem characteristics.

Problem characteristics and ASCM. ASCM incorporates a way for problem characteristics to influence the procedure chosen. In ASCM structural features of a particular problem are thought to influence which backup procedure will be used to solve it (Siegler & Shipley, 1995). To illustrate, the percentage of repeated addition, number series, and special-trick procedures used on problems is shown in Figure 2 as a function of operand families (see Campbell, 1994). In this presentation of the data each problem occurs twice (e.g., 7 x 8 appears in both the 7 operand and the 8 operand families). The use of many backup procedures was directly tied to problem characteristics. For example, for Grade 4 and 6 children, repeated addition was used primarily on problems with smaller operands, mainly operands of 2. Counting string was used mainly on problems with operands of 5. Finally, special tricks were used more frequently on large than small problems. For backup strategies there were strong connections between problem families and selection of procedures. These results support the conclusion that the selection of procedures was not random but systematically related to problem characteristics.

Problem-Size Effect and Use of Procedures

The purpose of the following analysis was to determine whether the problem-size effect is present when some procedures are used and not others, and if the overall problem-size effect is the result of averaging across different kinds of procedure trials. Specifically, we used product to predict solution latencies for all trials and retrieval trials only. The variance accounted for by product differs in Grade 4 ($R^2 = .43$ for all trials, .25 for retrieval trials) but not in Grade 6 (.47). Slope values differed substantially (77.6 and 23.3 in Grade 4, and 22.58 and 14.60 in Grade 6) (see Figure 3). Nevertheless, the product slope on retrieval trials is still statistically significant for both grades.

For Grade 4, the problem-size effect across all trials may be attributed at least in part to children's use of backup procedures on large problems (cf. LeFevre et al., 1996b). However, Grade 6 children used retrieval more frequently than Grade 4 children, meaning that the influence of backup procedures on all trials was reduced. As a result, for Grade 6 there was no difference between the variance accounted for by product on all trials versus retrieval trials only. In both grades the problem-size effect was not eliminated when only retrieval trials were considered. That the problem-size effect is evident on retrieval trials may result because the distribution of associations underlying the use of retrieval and the problem-size effect both are influenced by common factors, such as problem frequency and order of acquisition (Ashcraft & Christy, 1995; Campbell & Graham, 1985; Siegler, 1988a). These factors, and not problem-size per se, may be important in the developing of associations between problems and answers.

Differences between Grades 4 and 6 in the effectiveness of product in predicting latency on retrieval trials may reflect the influence that presentation frequency and use of procedures have on the associations between problems and answers. Associative measures may be better predictors than product in Grade 4 because they reflect the diverse use of procedures. Further, measures that influence the ability of product to predict solution latency, such as frequency of problem presentation, may not have strong influences on associations in Grade 4 because of lack of experience. However, as children gain experience, frequency may become more influential in determining the distribution of associations, allowing problem-size to appear more important because of its relation with this measure.

To examine whether frequency of problem presentation becomes more important, we calculated estimates of problem frequency based on frequencies for simple multiplication problems identified by Ashcraft and Christy (1995) in elementary mathematics textbooks. Frequencies for Grade 4 children were based on the number of times each problem was presented in Grade 3 and 4 textbooks. To reflect the cumulative influence of frequency of presentation on associations with experience, frequency estimates in Grade 6 were based on Grade 3, 4, 5, and 6 textbooks. Frequency of problem presentation was then used to predict solution latency. Frequency did account for solution latency in Grade 6 (r^2 = .42), but not for Grade 4 (r^2 = .01).

To determine whether the contribution of product is due to its relations with frequency, we contrasted the roles of product and percent retrieval in predicting latencies, after problem frequency was taken in account (see Table 6). According to ASCM, probability of retrieval-use reflects the distribution of associations between a problem and answers, and so probability of retrieval was used as the primary index of associative strength (see LeFevre et al., 1996a). In the first set of regressions, latency was first regressed on percent retrieval and problem frequency (Model 1) and then product was added to determine whether it made an independent contribution. Subsequently, latency was regressed on product and problem frequency (Model 2) and then percent retrieval was added to determine whether it made an independent contribution. In Grade 4 the probability of retrieval explained significant unique variance in latencies when entered after product and frequency, whereas product did not explain unique variance (see Table 6). For Grade 6, although it approached significance, percent retrieval did not account for unique variance when entered after product and frequency, and neither did product explain unique variance.

For Grade 4 the most important predictor is percent retrieval. Presumably, percent retrieval reflects the diverse procedures used and how accurately backup procedures are used. For Grade 6, the inclusion of problem frequency nullifies the unique influence of both product and percent retrieval. Theoretically, frequency is important in determining both the strength of associations and the problem-size effect. With development the associations between the problem and one particular answer become peaked. Product may become a better predictor because it is related to frequency, a variable that indexes this peaked structure.

Conceptual Understanding

Coding of Justifications

Children's responses on the conceptual tasks were scored in two ways. First, the number of correct answers was recorded separately for the comparison and the number sense tasks. Accuracy was an inappropriate measure for the remaining two tasks: Children were not required to give a final answer on the word problems task, and on the proofs task accuracy was redundant with the measure described below.

Second, a coding scheme was designed to categorize children's justifications on the comparison, number sense, word problems, and proofs tasks. The scheme was developed by analyzing the responses of a subset of children and identifying the different justifications they reported. Although similar justifications were reported on different tasks, justifications were coded separately across tasks. Descriptions of the justifications observed and scoring criteria are found in Appendix J. Justifications were placed into two categories based on the amount of conceptual knowledge of multiplication they reflected: justifications that reflected complete understanding of the underlying principles in question, and justifications that reflected partial or no understanding of the principles in question. The degree to which conceptual knowledge was demonstrated as reflected by the two categories was used for subsequent analyses of conceptual knowledge.

Comparison task. Children gave three justifications that reflected complete understanding of the principles assessed on the comparison task. The mean frequency of use across problems for all justifications is found in Table 7. For repeated-addition problems, children's responses were coded as application of repeated addition if they reflected the understanding that repeatedly adding a number is the same as multiplying that number by the number of times it was added. For commutativity problems, children's responses were coded as application of commutativity if they knew that the order in which two numbers are multiplied does not matter. For the related-fact problems, if children demonstrated that they understood the relations between two multiplication problems, then their responses were coded as a related-fact justification. Appendix J includes descriptions of all the justifications that did not reflect complete understanding of the important concepts.

Number Sense task. Frequency of use for the justifications is found in Table 8. For the operand-equal-to-product problems, one justification was judged to reflect complete conceptual knowledge. Children's responses were coded as the multiply-by-1 rule if they demonstrated knowledge that the product can only be equal to an operand if the other operand is one. For the operand-greater-than-product problems, the greater-than rule was judged as reflecting conceptual knowledge. Children demonstrated the understanding that in whole number multiplication an answer cannot be less than either the multiplicand or the multiplier. Finally, for estimation problems three justifications were identified as reflecting conceptual knowledge. Using multiplication, children reported that they multiplied all the digits in the problem together, and provided a correct answer, to see whether the given answer was correct[2]. Using partial multiplication, children reported multiplying some but not all of the digits in the problem and they did not provide a final answer to verify the accuracy of the given answer. Finally, some children justified their response with an ambiguous justification related to the appropriate magnitude of the answer or the appropriate direction of the operation given (ambiguous justification: direction or magnitude).

Word problems. For word problems, five justifications were judged to reflect conceptual understanding (see Table 9 for frequencies of all justifications used). For the irrelevant-information problem, children's responses were coded as correct multiplication if they applied multiplication to solve the word problem without using the irrelevant information provided. For the insufficient-information problem, some children realized that the problem did not contain the information needed to solve it (insufficient). If children correctly applied the multiplication, addition, and comparison steps (complete solution) on the multiplicative-compare problem, their response was considered conceptually based. Finally, for the Cartesian problem, two justifications were considered to reflect conceptual understanding. Children's responses were coded as correct multiplication if they appropriately used multiplication to solve the problem. Children's responses were coded as correct diagram if they used a diagram to demonstrate how the items in the problem could be grouped into sets.

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[2] In the instructions, children were specifically asked not to multiply the two numbers together. However, we coded the use of multiplication as conceptually based because by using it students demonstrated their knowledge of the appropriate direction of the operation and magnitude of the final answer.

Proofs. Conceptual understanding on the proofs task was reflected by a grouping procedure where children correctly divided beads into appropriate groups to demonstrate why an answer was correct (see Table 10 for all justifications used).

Reliability

An independent rater scored the responses from 30% of subjects, and reliability with the experimenter's scoring was calculated in two ways. First, a conservative reliability coefficient was calculated as the proportion of agreements between the two raters divided by the total number of agreements and disagreements. Second, an experimenter-specific reliability coefficient was calculated as the proportion of agreements between the two raters divided by the total number of agreements and disagreements on only the responses that the experimenter coded for a given problem. This was the probability that the second coder's judgment matched that given by the experimenter.

For the proofs problems raters agreed on all the justifications used. Inter-rater reliabilities for justifications used on the repeated-addition, commutativity, and word problems were also very high (see Table 11). Although not as high as the other problem types, there was also substantial agreement between the raters about justifications used on related-fact, operand-equal-to-product, operand-greater-than-product problems (see Table 11). Inter-rater reliabilities for justifications used on the estimation problems were considerably lower than those calculated for the other problem types, but there was still substantial agreement between raters (see Table 11). Any conclusions based on analyses of the number sense task estimation problems must be qualified by the somewhat lower reliability.

Changes in Conceptual Knowledge

To examine whether conceptual development in multiplication occurs during a period of time when emphasis is placed on memorization, the effects of grade on the proportion of conceptually based procedures were examined. Proportions were calculated using only justifications that reflected complete understanding of the concepts relevant for each problem. Proportions were calculated by dividing the number of conceptually based justifications for each individual on a specific problem type by the total number of problems of that type. Eight proportions of conceptual use were generated for each child, one for each problem type. Because two types of concepts were used to assess children's understanding of concepts important for multiplication, and changes in conceptual knowledge between Grades 4 and 6, two separate analyses of variance were carried out: one for concepts assessed using direct-application contexts, and one for concepts assessed using enabling contexts.

The first analysis included problems where concepts could be directly applied to a situation: proofs, repeated addition, commutativity, operand-equal-to-product, and operand-greater-than-product problems. Proportions were subjected to a 2(Grade) X 2(Sex) X 5(Problem Type) analysis of variance with repeated measures on the last variable. Use of conceptually based justifications varied as a function of sex and problem type. Across all problems, males used conceptually-based justifications somewhat more frequently than females (.85 vs .80), $F(1, 116) > 4.39$, ps$< .05$. The number of conceptually based justifications varied as a function of problem type, $F(4, 464) = 69.45$, $p < .001$. Mean proportions were .97, .97, .53, .69, and .96 for the repeated-addition, commutativity, operand-equal-to-product, operand-greater-than-product, and proofs problems respectively. Examination of simple effects revealed that children used fewer conceptually based justifications on the number sense problems than the other problems, Fs$(1, 348) > 70$, ps $< .01$. No changes between Grades 4 and 6 were evident. Thus on problems that required recall and application of a specific conceptual principles, children in Grade 4 were just as competent as those in Grade 6. However, ceiling effects are evident for the repeated-addition, commutativity and proofs problems, and so any conclusions regarding the development of concepts related to these problems must be tentative. If harder problems had been used to assess repeated addition, commutativity, and the ability to prove the answer, differences may have existed between the two grades. However, the absolute success of Grade 4 children on these problems is not trivial: Children displayed notable understanding of some concepts assessed in specific contexts.

The second analysis included problems where the relevant concepts enabled the generation of novel operations or inferences: related-fact, estimation, and word problems. Proportions were subjected to a 2(Grade) X 2(Sex) X 3(Problem Type) analysis of variance with repeated measures on the last variable. Use of conceptually based justifications varied as a function of sex and grade. Males used conceptually based justifications more frequently than females (.69 vs .58), $F(1, 116) = 7.00$, $p < .01$ and Grade 6 children used these justifications more than Grade 4 children (.71 vs .57), $F(1, 116)$ 11.88, $p < .01$. The number of conceptually based justifications did not vary as a function of problem type. Mean proportions were .64, .66, and .61 for the estimation, related fact, and word problems respectively.

Our results are consistent with the hypothesis that conceptual knowledge about multiplication develops unevenly. Generally, children displayed greater understanding on problems that involved direct application of a concept compared to enabling application. As well, on the problems that involve direct application of concept, there is no change from Grades 4 to 6. On related-fact, estimation, and word problems, where children were required to use previously learned knowledge in a new situation, performance improves markedly from Grades 4 to 6.

Accuracy and Proportion of Conceptual Use

Comparisons between accuracy and conceptual justifications are important for understanding the relation between the product of children's thinking (accuracy) and the underlying concepts they use to solve the problem (conceptual justification). Furthermore, this analysis allowed me to examine the effectiveness of product measures in assessment.

Proportions of accurate responses were examined with a 2(Grade) x 2(Sex) x 2(Task) analysis of variance with repeated measures on the last variable. Grade 4 children were less accurate than Grade 6 children on comparison and number sense tasks (.88 vs. .92 for comparison and .82 vs. .89 for number sense), and children generally were less accurate on the number sense task than on the comparison task (.85 vs. .90), Fs(1, 116) > 6.01, ps $< .01$.

The relations between percent correct and proportion conceptual use were examined for specific problem types. This analysis is based on two assumptions. First, principled verbal justifications imply conceptual understanding. Specifically, if children can explicitly explain or describe the principles or concepts important in a domain, then they are assumed to understand those principles or concepts; if they cannot, their understanding is assumed to be incomplete. Second, children can make correct responses without conceptual understanding. For example, children can solve problems in multiplication by rote memorization, without understanding the relevant concepts (McKenzie, 1993). The probability of a principled justification given a correct response for each of four types of problems in the comparison and number sense tasks was calculated for each subject. If there is a direct relation between the answer a child produces and the conceptual basis of that answer, the conditional probability should approach 1.0. If measures of product do not exactly represent the underlying process of thought, the conditional probability should be much less than 1.0. Repeated addition and commutativity problems were grouped together because children's responses were similar for these problems and they were both used to assess the direct application of concepts. Operand-equal-to-product and operand-greater-than-product were grouped together for the same reasons. Related-fact and estimation problems were also evaluated. Conditional probabilities for repeated addition/commutativity, related fact, operand-equal-to-product/operand-greater-than-product, and estimation problems were .98, .67, .71, and .75 for Grade 4 and .99, .74, .64, and .83 for Grade 6 children. Only on the repeated addition/commutativity problems does the probability approach 1.0. For the other problem types, many children were able to produce a correct answer without being able to justify that answer. Consequently, measures of performance that are based on whether a child is right or wrong may not reflect complete, explicit understanding of the domain.

The effects of grade, sex, and problem type on the proportion of conceptually based justifications on correct problems were examined using a 2(Grade) x 2(Sex) x 4(Problem type) analysis of variance with repeated measures on the last variable. Males used a greater proportion of conceptually based justifications than females (.85 vs. .73), $\underline{F}(1, 105) = 14.04$, $\underline{p} < .001$. Mean proportions were .98, .70, .67, and .79 for the repeated addition/commutativity, related fact, operand-equal-to-product/operand-greater-than-product and estimation problems respectively, $F(3, 315) = 13.78$, $p < .001$. Tests of simple effects revealed that on correct trials, children used more conceptually based justifications for repeated addition and commutativity problems than all other problems, and for estimation problems than for related fact or operand-equal-to-product and operand-greater-than-product problems, $Fs(1, 315) > 5.53$, $ps < .05$.

The effects of sex and task were qualified by a grade by sex by problem type interaction, $F(3, 315) = 3.27$, $p < .05$. There were no differences between boys and girls, at either grade, for the repeated addition and commutativity problems and the estimation problems. For related fact problems, there was no difference between Grade 4 boys and girls, but Grade 6 boys used more conceptually based justifications than girls did (.91 versus .57), $F(1, 420) = 12.22$, $p < .01$. For the operand-equal-to-product and operand-greater-than-product problems, Grade 4 boys used more conceptually based justifications on correct trials than girls (.88 versus .55), $F(1, 420) = 10.00$, $p < .01$, but Grade 6 boys and girls did not differ. There were no grade differences for any of the problem types, indicating there were no developmental changes in the relations between the answers children produce and their underlying conceptual knowledge.

Except for problems where children only needed to apply a previously learned principle, many children were able to produce an accurate answer but did not provide a conceptually based justification of that answer. Consequently, assessment instruments that rely solely on accuracy measures may not provide a complete characterization of children's thinking.

Working Memory

To evaluate age-related change in working memory, analysis of variance was used to compare backward digit and operation spans for children in Grades 4 and 6. Children's mean memory span increased from Grades 4 to 6 for the backward digit span (5.17 versus 6.43) and operation span (3.42 versus 4.63), Fs(1, 149) < 4.98, ps > .05. Both global and domain-specific working memory related to numbers and simple arithmetic increased with age. An increase in the ability to store and manipulate numbers and arithmetic information may be important for the development of arithmetic skill and individual differences.

Discussion

Computational Skill

Structural accounts and ASCM were used to explore the development of multiplication performance. Because self-reports were used to identify the procedures children used, data pertaining to reactivity and validity of the reports are discussed first. Next, four areas relevant to children's performance in multiplication are discussed: changes in the use of procedures, relations between the problem-size effect and problem characteristics, how use of procedures may be related to problem characteristics, and the relative effectiveness of structural versus associative indices in accounting for children's solution latencies.

Self-reports. The issue of whether self-reports of procedures are veridical and non-reactive is critical to the interpretation of the present data. To evaluate veridicality, self-reports are often compared to another source of data that reflect the processes involved. For example, self-reports could be compared to overt behavior as an index of use of procedures. Unfortunately, adults and older children rarely show evidence of overt calculation on simple arithmetic problems. Thus the veridicality of self-reports generated by older children must be evaluated based on the plausibility of the reports in relation to other, more objective measures (LeFevre et al., 1996a). To address the issue of veridicality, we compared latency and error data for each procedure. Because retrieval involves relatively few steps, we would expect that it would be the fastest and least error-prone procedure. We found that non-retrieval procedures (i.e., repeated-addition, number-series, and solutions based on special tricks) were slower than retrieval. As well, Grade 4 children were almost 100% accurate when they reported using retrieval, and were much less accurate when they used other non-retrieval procedures. Children's reports of procedure use were also consistent with the characteristics of the problems on which they were used. For example, children reported using repeated addition on smaller problems where this procedure would be effective. Thus, self-reports of procedures appear to be veridical.

In terms of reactivity, children in our study displayed similar errors and solved problems as quickly as children in research where no self-reports were used. However, they displayed fewer errors than previously reported. Three factors may account for the lower error rates. First, children in our study were required to solve only 28 multiplication problems as compared to 91-100 problems in previous studies. Our children may have been less fatigued and consequently had lower error rates. Second, procedural differences may have contributed to the lower error rates. In most of the previous studies problem presentation was controlled by the experimenter, usually within a specified time period. In our task, however, each trial was initiated by the children. Because children controlled when the next problem appeared, they may have been more prepared for each trial than children in previous studies. Finally, the requirement to verbalize performance may have affected processing (Russo et al., 1989). Specifically, use of self-reports may have caused children to attend to the accuracy of their answer more carefully than children in the other studies, resulting in lower error rates (Cooney & Ladd, 1992). If this hypothesis were correct, however, longer latencies might also be expected, but such was not the case. The influence of these different variables cannot be distinguished based on the present data.

Procedure use. Children in Grades 4 and 6 used multiple procedures to solve simple multiplication problems, and so an adequate model of children's performance in multiplication must account for use of multiple procedures. ASCM (Siegler & Shipley, 1995) includes an account of multiple procedure use. We found that changes in use of procedures between Grades 4 and 6 were consistent with predictions made in ASCM. The first prediction is that with increasing experience the distributions of associations should become peaked and consequently the use of retrieval should increase with age. Grade 4 children used retrieval on 67% of trials whereas Grade 6 children used retrieval on 88% of trials. Second, ASCM predicts that because information regarding the effectiveness of each procedure is stored in memory, the speed and accuracy with which each procedure was executed should increase with age. Grade 6 children were faster and more accurate than Grade 4 children on both retrieval and backup procedures. These results are consistent with the conclusion that development should not be viewed solely in terms of the acquisition of new, more effective procedures. Development also includes changes in the frequency of use of existing procedures and the refinement of those procedures (Lemaire & Siegler, 1995; Siegler, 1988a).

Problem-size effect and problem characteristics. There is no question that problems with larger numbers are solved more slowly that problems with smaller numbers. How to explain this phenomenon, and the importance of it, is less clear. Because structural variables successfully predict solution latencies, most models have emphasized problem size as an index of how retrieval occurs by means of spreading activation in the mental network. Without considering the multiple procedures children use, this emphasis may be misleading. Our results support the hypothesis that, although the problem-size effect is still evident when product is used to predict solution latencies, it decreases when problems with different characteristics are examined and different procedures are taken into account. For Grade 4 and 6 children, problems with a 2 or a 5 are solved much more quickly then would be expected based on problem size alone. We also found that whether a problem had a 2 or 5 in it was just as effective as product in predicting latencies. The use of product to demonstrate the appropriateness of a structural model of multiplication may not be warranted. The fact that different patterns of latencies are observed for different identifiable problem types suggests that children may be using different procedures selectively, depending on problem characteristics.

A striking finding was that children used procedures strategically on specific types of problems. Our results are consistent with predictions made by ASCM that problem characteristics are important in determining selection of procedures. For children in both Grades 4 and 6, repeated addition was used primarily on problems with smaller operands, counting string was used mainly on problems with operands of 5, and special tricks were used most frequently on problems with products greater than 40. My results support the conclusion that children in Grades 4 and 6 do not select procedures randomly but instead use problem characteristics to select procedures. Thus a complete account of the problem-size effect must include the strategic use of procedures for different problem types.

Use of multiple procedures and the problem-size effect. We found that percent retrieval was as good a predictor, if not a better predictor, of latencies than product for children in both grades. According to ASCM, percent retrieval indexes the underlying distributions of associations for problem-answer pairs and these associations are influenced by children's use of procedures. For Grade 4 children, percent retrieval accounted for much more of the variance in solution latencies than product. The strategic use of procedures in Grade 4 may be related to the observed problem-size effect. Procedures used on small problems may be faster and less error prone than those used on larger problems. For example, using repeated addition on small problems requires few steps, and hence there is less chance of error and faster solution times than if the procedure were used on a large problem. Use of special tricks on large problems involves first recalling previously learned information and then applying it to the present problem. Consequently, more time is required and there is a greater opportunity for errors to occur compared to use of repeated addition on small problems.

By Grade 6 structural and associative variables are equally predictive of solution latencies. How must the memory representation of number facts develop for product to increase in its predictiveness? Many researchers have argued that product is a surrogate of other, more psychologically plausible variables, such as frequency of problem presentation (Campbell & Graham, 1985; LeFevre et al., 1996b). Perhaps changes in the influence of problem frequency on associations and subsequently solution latencies are related to changes in the influence of product. Researchers have assumed that frequency of problem presentation should have the same influence across development (Siegler & Schrager, 1984). In this study, the influence of problem frequency changes between Grades 4 and 6. In Grade 4, frequency of problem presentation appears unimportant. Only use of procedures appears to influence the distribution of associations. If we consider multiplication to be a relatively novel task for Grade 4 children, the previous findings can be interpreted. Early in learning the important process for the acquisition of number facts is that associative connections must be made between problems and correct answers. This process should be more related to use of procedures than frequency of problem presentation because only the effective execution of backup procedures will strengthen associative connections between a problem and a correct answer. A problem can be presented numerous times, but if it is solved incorrectly, the association between the problem and correct answer will not be made. However, once initial associations have been made, then more frequent presentation of a problem should strengthen associative links because of the increasing success of retrieval. Consequently, frequency of problem presentation has a cumulative effect, so that by Grade 6, it is important for predicting solution latencies. Because small problems are presented more frequently than large problems, product appears to be a better predictor of solution latencies in Grade 6.

Conceptual Knowledge

Analyzing children's performance on a number of measures allowed me to observe many behaviors related to conceptual understanding and to examine the conditions that influence those behaviors. First, I found that children's understanding of concepts important for multiplication varied as a function of the context in which the concept was assessed. Children displayed better understanding of concepts more amenable to assessment involving direct application compared to concepts more amenable to assessment involving enabling applications. Second, changes between grades varied depending on the assessment context. Grade 6 children's performance was improved compared to that of Grade 4 children on problems where concepts were used to enable children to solve novel tasks, but not when concepts were directly applied to solve a problem. Finally, product (i.e. accuracy) measures of performance do not provide a complete account of children's performance.

Conceptual categories. The way in which conceptual knowledge is assessed can exercise a substantial influence on children's ability to demonstrate their understanding (Bisanz & LeFevre, 1992; Nunes & Bryant, 1995; Siegler, 1991). Based on evaluation of children' performance on the different types of problems, we found that children's understanding of mathematical concepts can vary depending on the concept being assessed. For example, on problems where children were required to directly apply core principles of multiplication, no differences existed in the performance of Grade 4 versus Grade 6 children. However, on related-fact, estimation, and word problems, where children were required to use previously learned knowledge in a new situation, older children's performance was better than younger children's. Because word problems involve understanding written language, perhaps changes in language comprehension account for differences between grades. However, on problems such as related-fact and estimation, which do not rely as heavily on language comprehension, Grade 6 children also did better than Grade 4 children.

Acquisition of core principles may be necessary for the development of procedures based on those principles, which can be generalized to novel settings. Children may have to understand a principle in a basic situation before they can understand it in another, novel setting (Nunes & Bryant, 1995). For example, understanding of principles may set an upper bound for generating procedures in novel or unstructured domains, because they influence and constrain the generation of new procedures (Dixon & Moore, 1996). How might core principles influence the construction of new procedures or adaptation of old procedure for novel tasks? Using a production system may help to understand the transition. A production system is a set of productions stored, presumably, in long-term memory. Productions are condition-action rules for changing the set of information currently activated in the cognitive system and they take the following form: "If X is present, then Y occurs." Bisanz and LeFevre (1992) argued that core principles could be represented in terms of productions (condition-action statements). These conceptual productions could be modified by knowledge-acquisition productions to create new procedures in novel settings. For example, a production for the principle of repeated addition takes the form "If $W + W + W + W = Y$ and $4 \times W = Y$, then $W + W + W + W = 4 \times W$." A production for the principle of commutativity takes the form "If $W \times Y = Z$ and $Y \times W = Z$, then $W \times Y = Y \times W$." These production statements can be modified by a knowledge-acquisition productions to create knowledge about part-whole relations and related-fact procedures: "If $W + W + W + W = 4 \times W$, and $W \times Y = Y \times W$, then (A) numbers can be divided into parts, and (B) there are different ways to derive a number. If A and B, then numbers are compositions of other numbers and known number facts can be broken down to derive the solution to unknown combinations."

With potential relations between basic principles and the construction of novel procedures made explicit through productions, in future research we could examine the development of conceptual understanding in multiplication through a training study. Specifically, children with little experience in multiplication could be taught the principles of repeated addition and commutativity during a number of sessions. In each session their understanding of part-whole relations and related-facts could be assessed to determine if training in basic principles led to the construction of novel procedures.

Accuracy and conceptual justifications. Verbal justifications were used to evaluate the relations between the product of children's thinking (accuracy) and conceptual understanding. The use of verbal justifications was based on the assumption that they reflect conceptual understanding. However, requiring verbal justifications to demonstrate conceptual understanding may underestimate children's knowledge. For example, children with language difficulties may have conceptual understanding in multiplication, but be unable to express that understanding verbally. Even though verbal justifications may lead to underestimating children's understanding, they still provide valuable information. Specifically, if children can explicitly explain or describe the principles or concepts important in a domain, then we can be relatively certain they understand those principles or concepts. We found that many children in both grades were able to produce an accurate answer but did not provide a conceptually based justification of that answer, especially on problems requiring the application of previously learned knowledge to novel settings. This discrepancy could be the result of two factors. First, many students may have provided a correct answers on problems through a rote procedure or incomplete knowledge, even if they did not have a full understanding of the important concepts. Second, children may have provided correct answers because they understood the important concept, even though they could not explain the concept. Likely, different children fell into either of these two categories. The purpose of this study was not to determine whether verbal justifications reveal complete understanding and we do not argue that verbal justifications should replace accuracy and other measures. Rather, we conclude that assessment instruments that rely solely on accuracy measures may not provide a complete characterization of children's thinking. Multiple measures, including verbal justifications, are required to evaluate conceptual understanding.

Working Memory

Age-related changes in working memory involves children's ability to process and store increasingly large strings of information with development. We found that Grade 6 children can remember more digits on the backward digit span, and remember more answers to multiplication problems on the operation span, than Grade 4 children.

Individual Differences and Mathematical Achievement

Computational Skill and Conceptual Understanding

Use of procedures

Relations between children's performance on the WRAT-III and on the computational task were analyzed separately for each grade. In Grade 4, participants who did well on the WRAT-III were more likely to use retrieval on the computational task, $r(58) = .49$, $p < .01$, and less likely to use relatively inefficient procedures such as counting string, $r(58) = -.31$, $p < .02$, and repeated addition, $r(58) = -.42$, $p < .01$. These results support the view that selection of procedures is related to individual differences in skill. As well, for Grade 4 mathematics achievement was associated negatively with solution latency, $r(58) = -.57$, $p < .01$ and positively with accuracy, $r(58) = .62$, $p < .01$. In contrast to Grade 4, in

Grade 6 mathematics achievement was not associated with proportion use of retrieval, counting string, or repeated addition, perhaps because there was little variability among children in terms of procedures (retrieval was used on 88% of the problems by Grade 6 children). The only significant association was between achievement and solution latency, $r(58) = -.35$, $p < .01$. In terms of procedural skills, the efficient use of procedures by skilled Grade 4 children may lead to problem-answer associations that are highly connected, leading to faster retrieval in Grade 6.

Conceptual Understanding

Relations between mathematics achievement and conceptual knowledge on each problem type were analyzed separately for each grade. The proofs task was not included because of restricted range in variability. In Grade 4, high achievement on the WRAT-III arithmetic subtest was related to increased use of conceptually based justifications on all problems (see Table 12). Achievement was not related to use of conceptually-based justifications in Grade 6. Similar to the findings with the computational task, perhaps early understanding of concepts facilitates skilled children in developing broader problem-solving skills.

Working Memory

Results are reported in three sections. First, to evaluate the underlying components of the operation span, backward digit span and measures of multiplication processing were used to predict individual differences on the operation span. Second, operation span and backward digit span were correlated with simple measures of processing skill in simple multiplication to determine whether working memory is related to mathematics. Third, memory span tasks were used to predict individual differences in mathematics achievement after measures of multiplication processing had been taken into account in order to evaluate whether domain specific processes in working memory are related to mathematics.

Underlying Components of the Operation span

For each grade, score on the backward digit span, mean solution latency on correct multiplication problems on the computational task and proportion retrieval use on the computational task were used to predict individual differences on operation span. Solution latency and proportion retrieval are correlated, and solution latency has a greater correlation with operation span than percent retrieval for both grades (see Table 12). Consequently, of these two tasks, solution latency was always entered first in regression analyses. First, score on backward digit span was regressed on operation span, and then solution latency and percentage retrieval were added to determine whether they made an independent contribution (Model 1). Subsequently, solution latency and percentage retrieval were entered followed by score on the backward digit span (Model 2). As shown in Table 13 both backward digit span and solution latency accounted for significant variance in the score on operation span for children in Grade 4. As well, backward digit span and solution latency added unique variance when they were the second variable added. For Grade 6 only solution latency explained significant variance. Proportion retrieval did not contribute any unique variance after solution latency was taken into account for either grade.

For Grade 4 children, operation span appears to involve global processing in the central executive because backward digit span accounts for significant variance. As well, domain-specific processing, mainly speed of processing simple multiplication problems is also involved. In Grade 4, it may be that global and domain-specific processing are involved in the operation span task because the domain is not clearly specified (i.e., children have not consolidated multiplication facts in memory). By Grade 6, efficiency in mathematics processing is more important for accounting for individual differences in the operation span than global working memory. As children's multiplication knowledge develops, assessment of working memory must include domain specific processing because of the increased experience in the domain. A substantial portion of variance was not accounted for in both grades by the backward digit span and multiplication processing measures. Perhaps another factor is important for performance on this measure, such as a strategic process not accounted for by the backward digit span (e.g., putting the answers in numerical order).

Relations Between Working Memory and Arithmetic

Relations between children's performance on the span tasks and measures of mathematics achievement and processing were analyzed separately for each grade. In Grade 4, children with larger backward digit span, and children with larger operation span solved simple multiplication problems faster, used more retrieval on simple multiplication problems, and had higher scores in mathematics achievement (see Table 12). These results are consistent with the hypothesis that global processing in the central executive, storage, and domain-specific processing are related to mathematics achievement and simple mathematical processing in Grade 4.

For Grade 6, children with larger backward digit span and children with larger operation spans solved simple multiplication problems faster, and used more retrieval on simple multiplication problems (see Table 12). Of the two memory tasks, only backward digit span was related to mathematics achievement in Grade 6 (see Table 12). We know from our previous results that by Grade 6, operation span is related to working memory processing specific to multiplication. Because tests of mathematics achievement like the WRAT-III involve skills other than multiplication, more global measures of working memory may be more directly related to tests of mathematics achievement.

Based on the task analyses I argue that backward digit span reflects global processing and storage for children in both grades. The importance of global working memory, as indexed by the backward digit span, remains consistent from Grades 4 to 6 in relation to mathematics achievement. The ability to hold and manipulate numbers in memory remains important for simple multiplication and mathematics achievement regardless of age. Changes in the relations between measures of arithmetic and operation span require more precise interpretation because earlier we found that the processes underlying the operation span may change with age. Specifically, in Grade 4, the operation span involves both global and domain specific processing. By Grade 6, only domain-specific processing is important. As the operation span progressively reflects domain-specific processing (i.e., efficient multiplication fact retrieval), it changes from being important to a broad range of mathematics skills to being related to just multiplication performance in Grade 6. Consequently, even within mathematics the efficiency with which the central executive executes task-specific processes is important in understanding the relations between arithmetic and working memory. Again, any conclusions regarding age-related changes must be tentative because they are made on the basis of comparing correlations for two different groups.

Components of Working Memory and Mathematics Achievement

To determine whether mathematics-specific processes account for performance better than general processes, operation span and backward digit span were used to predict individual differences in mathematics achievement. First, I entered the score on backward digit span, then operation span (Model 1). Second, the order of entry for the variables was reversed (Model 2). For Grade 4, both backward digit span and operation span entered first explained significant variance in the score on the WRAT-III (8% versus 10%), Fs > 5.06, ps < .05. Neither of these variables explained significant unique variance in mathematics achievement when entered second (see Table 14). For Grade 6, only backward digit span explained significant variance in the score on the WRAT-III, F = 11.96, p < .01 (see Table 14). These results are consistent with the hypothesis that global working memory processing becomes increasingly important in predicting general mathematics achievement with development.

To determine the best mathematics efficiency predictor, solution latency and proportion retrieval were used to predict WRAT-III. In the first step, we entered solution latency and then proportion retrieval (Model 1). In the second model the order was reversed (Model 2). For Grade 4, both solution latency and proportion retrieval entered first explained significant variance on the WRAT-III (33% versus 24%), Fs > 16.54, ps < .01. Only solution latency explained significant unique variance in mathematics achievement when entered second (see Table 15). For Grade 6 children, only solution latency entered first explained significant variance in the score on the WRAT-III (13% versus .004%), Fs >12.78, ps < .01. It appears that solution latency, which presumably reflects speed of processing and acquisition history, is more important than specific acquisition history as measured by percentage retrieval. Consequently, solution latency was used to predict WRAT-III scores in subsequent analyses.

For Grade 4 solution latency and operation span were used to predict WRAT-III. In the first model I entered solution latency and then operation span (Model 1). In the second model the order was reversed (Model 2). Solution latency accounts for a larger amount of variance in mathematics achievement compared to operation span (33% versus 10%). Solution latency explained significant unique variance in mathematics achievement when entered after operation span whereas operation span did not (see Table 16). For Grade 6, backward digit span and solution latency were used to predict WRAT-III scores. In the first model I entered solution latency then backward digit span (Model 1). In the second model the order was reversed (Model 2). For Grade 6, backward digit span accounts for a larger amount of variance than solution latency (17% versus 13%).

Both solution latency and backward digit span explained significant unique variance in mathematics achievement when entered after the other (see Table 16).

Relations Among Measures

Correlations are presented in Table 12. Many of the relations between specific variables have been discussed previously. Consequently, in the present discussion only the main patterns of relations are highlighted. First, the correlations among the measures are discussed and then results from the cluster analysis for each grade.

Generally, individual differences in the performance of Grade 4 students were highly related among the measures. For example, individual differences in raw scores on the WRAT-III were related to performance on all other measures. The specific nature of these relations for computational, conceptual, and working memory measures has been discussed previously in this paper. Further, working memory and computational measures were significantly related. Finally, individual differences on some of the conceptual measures (i.e., Equal-to/Greater-Than, Estimation, and Word Problems) were related to variation on the computational and working memory measures. Hierarchical cluster analysis revealed two clear clusters. Specifically, a computational/achievement cluster was evident which included median latency for solving multiplication problems, the percent of retrieval use, and raw score on the WRAT-III. A working memory cluster consisting of the operation span and backward digit span was also present. Measures of conceptual understanding were not clearly part of any cluster.

In contrast to Grade 4, few significant relations among the measures were present for students in Grade 6. Individual differences in the raw score on the WRAT-III are only related to performance on the backward digit span and median latency for solving simple multiplication problems. The only other significant relations in individual differences present are between (a) operation span, median latency, and proportion retrieval, (b) operation span and backward digit span, and (c) related-fact and word problems. In terms of cluster analysis, two clear clusters are evident. First, raw score on the WRAT-III and backward digit span form a cluster. Added to this cluster in later steps are median latency and operation span. A second clustering of measures includes the related-fact and word problems.

Discussion

Computational Skill and Conceptual Knowledge

The changing relations between mathematics achievement and multiplication skill were similar when computational skill and conceptual understanding were evaluated. In Grade 4, low achievement was associated with use of immature procedures and ineffective use of procedures. By Grade 6, however, only slow solution latency was associated with low achievement. Perhaps the early efficient use of procedures by higher-achieving children influences the developing problem-answer associations so that they are better connected and consequently retrieved faster later on. Conversely, low-achieving children's use of a greater mix of immature strategies may also influence the developing connections, but in a different way. Specifically, the distribution of associations for individual problems for these children would be flat, leading to less retrieval. Although, they eventually may use retrieval, their distributions still may not be as peaked as those for higher-achieving children. Consequently low-achieving Grade 6 children may attempt to retrieve more times before their confidence interval is exceeded, leading to longer latencies.

Consistent with the changes in individual differences observed using computational measures, conceptual ability is related to standard achievement measures in younger children but not in older ones. Perhaps the early understanding of concepts provides skilled children with a "head start" in multiplication that allows them to practice problem-solving skills for a longer period of time. By Grade 6, even though all children understand the basic concepts of multiplication, skilled children may demonstrate more advanced problem-solving abilities because of the greater amount of experience using concepts. Any hypotheses related to computational and conceptual knowledge must be tentative, however, pending longitudinal evidence showing that the low-skilled children in Grade 4 would still be the low-skilled children in Grade 6.

Working Memory

We found that working memory is related to achievement in mathematics. Our results differ somewhat from the existing literature on working memory and performance on complex tasks. Two positions are currently held regarding the effectiveness of working memory measures in predicting individual differences on complex tasks: (a) measures of global working memory are more effective than domain-specific measures (Engle & Turner, 1989), and (b) domain-specific measures are more effective than global measures (Daneman & Carpenter, 1983; LeFevre, 1993). I found that both global and domain specific working memory can account for individual differences, depending on age and what aspect of a complex skill that is measured.

Different working memory processes accounted for mathematics achievement in Grade 4 compared to Grade 6 children. For Grade 4 children, processing speed of multiplication facts is the most important predictor of mathematics achievement. Because operation span does account for some variance on its own, it would appear that working memory processes involving specific mathematics capabilities is also important. Consequently, efficiency of processing of mathematics-specific information by the central executive was most important for mathematics achievement in Grade 4. By Grade 6 however, general processing in the central executive was most important.

This difference may be due to the nature of the problems solved by the children on the WRAT-III. The problems solved by the Grade 4 children involved simple and multi-digit arithmetic problems. Past learning is important for these problems, and speed of processing as measured by solution latency may reflect this past learning. Although the Grade 6 children solved these problems, most of them also solved more difficult problems that required more novel problem solving skill such as algebra and fractions. Perhaps general working memory is more important for these novel problems. Another reason for why solution latency was more related to achievement in Grade 4 compared to Grade 6 was the restricted range of latencies observed in Grade 6. For Grade 4 there was large variability in solution latency for correct multiplication problems (S.D.= 1139 ms). By Grade 6, the variability was much smaller (465 ms), resulting deflated correlations with other variables.

Age related change in working memory may be the result of increases in total space (storage and processing space), or a redistribution of space between storage and operating components. Case et al. (1982) present evidence supporting the latter. They argued that as children become faster and more efficient in processing information, less processing space is required and more space becomes available for storage. As a result, longer strings of information can be remembered. The present results are consistent with this hypothesis. Specifically, I found that measures of processing speed (solution latency) become more important in predicting individual differences on operation span from Grades 4 to 6. As performance on operation span increases, so does the relative importance of processing speed in predicting performance.

Relations Among Measures

Based on the patterns of correlations and cluster analysis, in Grade 4 individual differences in computational knowledge, conceptual knowledge, and working memory are all related. For students in Grade 6, fewer relations among different areas of mathematical knowledge are present; mathematics functioning is compartmentalized into computational/working memory and conceptual domains. The finding that conceptual knowledge is related to computational skills in Grade 4 but not in Grade 6 may be due the fact that conceptual knowledge is important in the development of computational skills (Baroody, 1995; Geary et al., 1992). In Study 1 it was found that computational skills in multiplication are forming in Grade 4 and are established by Grade 6. In Grade 4, to develop and use repeated addition and derived fact procedures to solve simple multiplication problems, students need to understand part-whole relations. However, by Grade 6, students have well-developed procedures and do not need to rely on conceptual principles when solving simple multiplication problems.

Based on these results it appears that although standardized achievement tests, such as the WRAT-III, do reflect individual differences in children's computation abilities, they are less effective in assessing conceptual knowledge, especially in later grades. This shortcoming of standardized achievement tests has important implications for assessing children's performance. If assessment of mathematics skills is based solely on standardized achievement tests that only measure computational skills, researchers and clinicians will lose valuable information about children's understanding.

GENERAL DISCUSSION

The present results on age-related change and individual differences in multiplication have implications for four areas important to providing an integrated account of the acquisition of mathematical skill and knowledge. First, any inquiry into the acquisition of skill or knowledge requires careful consideration of the methods employed for assessing children's performance. Implications from the present findings for the assessment of children's performance in multiplication are addressed. Second, clearly defined models of multiplication are needed to interpret data gathered through assessment. Implications from the present results are discussed in terms of models of performance in multiplication. Third, an integrated account of the acquisition of multiplication skill and knowledge involves identifying principles of development common to the different areas involved in multiplication. Consequently, implications of the current findings are discussed in terms of principles of development. Finally, changing patterns from younger to older children in the relations between measures can be used to make inferences regarding the development course in the acquistion of multiplication skill and knowledge.

Implications for Assessment

A core issue for both empirical and clinical work is the assessment of children's skill and knowledge. How performance is assessed has a substantial impact on the models of knowledge acquistion and development. Based on the present findings, the use of multiple measures appears essential for a complete understanding of children's acquistion of multiplication skill and knowledge. Evidence for the necessity of multiple measures is found in three areas: self-reports of procedure use, verbal justification and conceptual understanding, and limitations of standardized achievement measures.

Self-Reports

Multiple measures are necessary for providing converging evidence. The assessment of simple arithmetic in children has typically involved accuracy data, solution latencies, and behavioral observations. I found immediately retrospective self-reports provided valuable information regarding the diversity of procedures children use to solve problems. The use of self-reports helped to distinguish between procedures such as repeated addition and counting string, that would appear the same based on behavior data alone. Although self-reports provide detailed information, it would be unwise to use them without other measures of performance. Other concurrent measures are required to evaluate the veridicality and reactivity of self-reports. To evaluate veridicality, self-reports were compared to latency and error data for each procedure. For example, because retrieval involves relatively few steps, it would be expected to be the fastest and least error-prone procedure. In fact non-retrieval procedures (i.e., repeated-addition, number-series, and solutions based on special tricks) were slower than retrieval. As well, Grade 4 children were almost 100% accurate when they reported using retrieval, and were much less accurate when they used other non-retrieval procedures.

Concurrent accuracy and latency data were also required to evaluate the reactivity of self-reports. Although children in this study displayed similar errors and solved problems as quickly as children in research where no self-reports were used, they displayed fewer errors than previously reported. Differences in methods between the other studies and this one may account for these findings. However, it also may be that use of self-reports caused children to change their behavior. As long as concurrent measures are used to evaluate self-reports, the added information they provide warrants their use.

Verbal Justifications and Conceptual Understanding

Using a number of measures provides additional information that may go unnoticed if only a single measure is used. For example, children are considered to understand a concept if they produce the correct answer to problems related to the concept. Using measures of accuracy and verbal justifications, I found that many Grade 4 and 6 children were able to produce a correct answer, but could not provide a conceptually based justification for that answer. Some children who answered correctly may have understood the concept, even though they could not explain it. However, many children may have used a rote procedure to solve the problem and had little understanding of the concept involved. Assessment instruments that rely solely on accuracy measures may not provide a complete characterization of children's thinking. Multiple measures, including verbal justifications, are required to evaluate conceptual understanding.

Limitations of Standardized Achievement Measures

Multiple measures are need to provide a complete account of children's skills and knowledge. Standardized achievement tests are used extensively in educational, clinical, and research settings as the sole measure of mathematical ability. Many of these tests, like the WRAT-3, involve only mathematical computations and may not reflect all areas of mathematical functioning. By using multiple measures of multiplication and examining relations with the WRAT-3, the utility of the achievement tests for assessing all areas important to simple arithmetic was evaluated. Although standardized achievement tests, such and the WRAT-III, do measure children's computation abilities, they are less effective in assessing conceptual knowledge, especially in later grades. If assessment of mathematics skills is based solely on standardized achievement tests that only measure computational skills, researchers and clinicians will lose valuable information about children's understanding.

Implications for Cognitive Models of Multiplication

A reciprocal relationship exists between empirical data and theoretical models of performance. Models are necessary to organize and interpret empirical data. In the Results and Discussion sections theoretical models were used to interpret data. However, empirical data are important for developing and shaping models of performance. For models to remain relevant they must be flexible enough to account for and incorporate new data. Implications of the present data for models of computational skills, conceptual knowledge, and working memory are discussed.

Computational Skills

Based on empirical data gathered on Grade 4 and 6 children's performance on simple multiplication problems, models of computational skill must account for the influence of problem characteristics on performance and the strategic use of procedures on different problems. First, different patterns of solution latencies were evident for different identifiable types of problems. As well, whether a problem had a 2 or a 5 in it was just as effective in predicting solution latencies as a structural variable like product. Children were also observed to use procedures strategically on specific types of problems. For example, repeated addition was used primarily on problems with smaller operands, and special tricks were used only on problems with operands greater than forty. Any account of children's performance in multiplication must include how problem characteristics are important in determining solution latency, and the strategic use of procedures. A model that is promising in this regard is ASCM. Although this model accounts for why a new procedure may be used, and why after a procedure is used it is applied more frequently to specific problem types, it does not account for why a specific kind of procedure may be used initially on some kinds of problems but not others. Further research is needed on the processes by which children use a procedure for the first time, on specific types of problems.

Models of Conceptual Understanding

Researchers and educators often assume that children either have conceptual understanding in a domain or they do not (Greeno, 1983). I found that children's understanding of concepts important for multiplication can vary depending on the particular concept, and how the concept is assessed. Specifically, children's understanding of concepts that could be directly applied to solve a problem was greater than those concepts used to enable the generation of novel knowledge. Nunes and Bryant (1995) found that children's understanding of commutativity in multiplication can vary from one situation to another. One way that appears to be useful for promoting conceptual knowledge is to ask children to come up with as many ways as possible to solve a particular problem (Sweller, Mawer, & Ward, 1983). Consider a child's understanding of counting. A good understanding of counting may be demonstrated when the child knows that counting can occur from left to right, from right to left, or haphazardly and, as long as all of the items are counted, still yield the same answer (Geary, 1994). Hence, models of conceptual understanding must account for how children can understand some important concepts but not others.

Working Memory

Models of working memory and of the influence of working memory on skilled performance focus on global versus domain-specific memory processes. I found that both global and domain specific working memory can account for individual differences, depending on age and what aspect of a complex skill that is measured. Different working memory processes accounted for mathematics achievement in Grade 4 compared to Grade 6 children. For Grade 4 children, processing speed of multiplication facts is the most important predictor of mathematics achievement. By Grade 6, however, general processing in the central executive was most important. Hence models of working memory need to incorporate both global and domain-specific processes. Further, models need to be useful for determining when specific versus global processes are used on different types of tasks and under what conditions different memory processes are employed.

Implications for Principles of Development

The use of multiple measures allowed the opportunity to provide an integrated account of the development of multiplication not present in the existing literature. By using measures of computational skills, conceptual knowledge, and working memory, I was able to identify a principle of mathematical development common to all areas.

Variability Within Ages

A characteristic of children's performance was variability: Children in Grades 4 and 6 used various procedures to solve simple multiplication problems, and their understanding of concepts important for multiplication varied depending on the context. Development is often described in terms of a progression through stages where a single procedure or way of thinking predominates. As children move from one stage to the next, more advanced procedures replace less advanced ones. My results are inconsistent with this view of development.

There is no question that young children use multiple procedures to solve cognitive tasks (Lemaire & Siegler, 1995; Siegler, 1987, 1988a, 1988b, 1989; Siegler & McGilly, 1989). However, variability in use of procedures is not limited to young children. Older children used multiple procedures to solve multiplication problems even during a period of time when heavy emphasis is placed on memorization. Adults also use multiple procedures on simple tasks (LeFevre et al., 1996a; LeFevre et al., 1996b; Siegler & Lemaire, 1997). Variability in use of procedures can no longer be viewed as reflecting immature performance, while use of a single procedure is considered to be the norm for advanced processing. My results support the hypothesis that development involves not only the addition of new procedures, but changes in how often individual procedures are used and also improved execution of procedures (Lemaire & Siegler, 1995). Having a variety of procedures to use allows children to apply procedures adaptively. Depending on the task demands or problem characteristics, they can apply the most effective knowledge from their repertoire.

Conceptual development is also often considered to occur in an all-or-none fashion: Older children understand multiplication, younger children do not. However, because many forms of behavior are related to understanding, and understanding can vary depending on the situation, conceptual development cannot be viewed in absolute terms.

Based on the present results it appears that conceptual development may involve the uneven advancement of concepts, with a child being in different "stages" depending upon the specific concept being evaluated. Specifically, Grade 4 and Grade 6 students' understanding of multiplication was not substantially different. For example, there were no differences in their understanding of concepts important for the direct application of a principle. Further, many students in Grades 4 and 6 demonstrated understanding of enabling concepts, although more students in Grade 6 understood these concepts.

Development of simple multiplication skill and knowledge does not involve a uniform progression through stages where one type of thinking or procedure dominates. Rather, development appears to be uneven and variable, with children at many ages demonstrating use of similar procedures and ways of thinking. What changes is the frequency with which different procedures are used, or ways of thinking are applied. These conclusions are limited to development in simple arithmetic. It would be important to determine if the principle of variability applies in the development of more complex skills.

Changing Relations Among Measures

Computational skill, conceptual knowledge, and working memory were all related to individual differences in achievement. However, the pattern of the relations was different in Grade 4 compared to Grade 6. By examining theses different patterns of individual differences, inferences can be made about how development must occur in order to observe the change in patterns from Grades 4 to 6. For example, examination of the changing relations between achievement and computational skills, and achievement and conceptual understanding provides information from which inferences can be made regarding the development of mathematical achievement. In Grade 4, low achievement was associated with use of immature procedures and ineffective use of procedures. By Grade 6, however, only slow solution latency was associated with low achievement. Similarly, conceptual knowledge was related to standard achievement measures in younger children but not in older ones. Perhaps the early efficient use of procedures and understanding of concepts facilitates the quick acquisition of multiplication facts, resulting in increased achievement levels for children in Grade 4. This may also lead to a longer period to practice multiplication facts for high achieving children, resulting in increased speed of retrieval being related to achievement in Grade 6. Although these analyses are speculative, they point to the influence of early experience on mathematical achievement.

Different patterns in the relations among all the measures were observed for students in Grade 4 versus students in Grade 6. I found that mathematical knowledge is relatively homogeneous for students in Grade 4. Computational knowledge, conceptual knowledge, and working memory are all related. For students in Grade 6, fewer relations among different areas of mathematical knowledge are present; mathematics functioning is compartmentalized into computational/working memory and conceptual domains. A problem with the analyses used in the present study is that individual differences in two different grades were used to make inferences regarding development. While these inferences can guide future research, they are only speculative. An effective evaluation of individual differences and developmental outcomes must involve the longitudinal study of a number of individuals.

Table 1

Error Rates (Percentage) on Single-Digit Multiplication Problems in Six Studies

Grade	Present Study	Siegler (1988a)	Cooney et al. (1988)	Campbell & Graham (1985)	Lemaire & Siegler (1995)	LeFevre et al. (1996a)
2	-	-	-	-	55.0	-
3	-	29.8	17.1	23.1	-	-
4	9.0	-	13.7	25.3	-	-
5	-	-	-	16.8	-	-
6	1.6	-	-	-	-	-
Adults	-	-	-	7.7	-	3.0

Table 2

Percentage of Errors as a Function of Type on Single-Digit Multiplication Problems in Two Studies

	Present Study		Campbell & Graham (1985)			
Errors	Grade 4	Grade 6	Grade 3	Grade 4	Grade 5	Adults
Operand Related	41	61	49	48	69	79
Operand Unrelated	16	13	21	22	14	14
Non-table	27	19	30	31	17	7
Out of Time	16	7	-	-	-	-

Table 3

Mean Solution Latencies (in seconds) on Single Digit Multiplication Problems in Five Studies

Grade	Present Study	Siegler (1988a)	Campbell & Graham (1985)	Lemaire & Siegler (1995)	LeFevre et al. (1996a)
2	-	-	-	9.90	-
3	-	10.77	2.84	-	-
4	2.56	-	3.64	-	-
5	-	-	1.87	-	-
6	1.53	-	-	-	-
Adults	-	-	.83	-	1.34

Table 4

Procedure use as a Function of Grade for Single-Digit Multiplication Problems

	Grade 4				Grade 6			
Procedure	Overall Use (%)	Use Once[a] (%)	Latency[b] (ms)	Accuracy[c] (%)	Overall Use (%)	Use Once[a] (%)	Latency[b] (ms)	Accuracy[c] (%)
Retrieval	67	98	2191	96	88	100	1469	99
Special Trick	16	77	5693	83	8	43	3738	98
Repeated Addition	8	37	4783	88	2	23	2796	95
Counting String	5	30	3792	83	1	10	1525	100
Other	3	42	-[d]	17	0.42	10	-[d]	0

[a] Percentage of children who used the procedure at least one time.

[b] Mean of median latencies for correct trials computed for children using the procedure at least two times.

[c] Percentage correct for children using the procedure at least two times.

[d] Insufficient data for computing a median latency.

Table 5

Summary of R^2 change values for Latencies Regressed on Product and Twos/Fives Variables

Order of Entry	Grade 4	Grade 6
Model 1		
Product	.43*	.47*
Twos/Fives	.31*	.22*
Model 2		
Twos/Fives	.66*	.55*
Product	.08	.14*

*$p < .05$

Table 6

Summary of R^2 change values for Solution Latencies Regressed on Problem Frequency, Product and Percentage Retrieval

Order of Entry of Variables	Grade 4	Grade 6
Model 1		
Percent Retrieval and Problem Frequency	.75*	.54*
Product	.03	.06
Model 2		
Product and Problem Frequency	.49*	.52*
Percent Retrieval	.29*	.08

*$p < .05$

Table 7

Frequency of Justification Use for Comparison Problems

Problem Type	Justification	Percent Use: Grade 4	Percent Use: Grade 6
Repeated Addition			
	Application	96.7	98.0
	No Application	3.3	1.7
	Computation	0.0	0.0
Commutativity			
	Application	96.4	98.0
	No Application	2.8	1.0
Related Fact			
	Application	60.8	72.0
	Direction or Magnitude	19.2	11.0
	No Application	16.7	15.0
	Computation	1.7	0.4

Table 8

Frequency of Justification Use for Number Sense Problems

Problem Type	Justification	Percent Use Grade 4	Percent Use Grade 6
Operand = Product			
	Multiply By 1	51.1	54.0
	Direction or Magnitude	42.8	43.0
	None	5.0	1.1
Operand > Product			
	Greater than Rule	66.1	73.0
	Direction	20.5	26.0
	None	12.2	0.0
Estimation			
	Multiplication	15.6	31.0
	Partial Multiplication	20.6	18.0
	Ambiguous Justification:	23.9	21.0
	Direction or Magnitude		
	Ambiguous Justification	11.7	11.0
	None	22.2	14.0

Table 9

Frequency of Justification Use for Word Problems

Problem Type	Justification	Percent Use	
		Grade 4	Grade 6
Irrelevant Information			
	Correct Multiplication	60.0	75.0
	Irrelevant Computation	31.7	23.3
	Addition	6.6	0.0
Insufficient Information			
	Insufficient	35.0	68.3
	Unsure	10.0	5.0
	Inappropriate Multiplication	45.0	20.0
	Addition	10.0	3.3
	Irrelevant Computation	0.0	1.7
Multiplicative Compare			
	Complete Solution	58.3	78.3
	Incomplete Solution - Addition	6.7	1.7
	Incomplete Solution - Compare	15.0	13.3
	Incomplete Solution - Extra Addition	10.0	3.3
		8.3	0.0

Table 9 continued

Frequency of Justification Use for Word Problems

Problem Type	Justification	Percent Use Grade 4	Percent Use Grade 6
Cartesian			
	Correct Multiplication	41.7	48.3
	Correct Diagram	3.3	15.0
	Incorrect Matching	26.0	33.3
	Incomplete Solution -Extra	0.0	0.0
	Addition	6.7	1.7

Table 10

Frequency of Justifications Use for Proofs

Problem Type	Justification	Percent Use Grade 4	Percent Use Grade 6
Proofs			
	Grouping	93.3	100.0
	Inappropriate Grouping	6.7	0.0

Table 11

Inter-Rater Reliability for Individual Justifications on Conceptual Tasks

Problem Type	Justification[a]	Number of Observations[b]	Conservative Reliability	Rater-Specific Reliability
Repeated Addition	All	248	99.2%	100.0%
Commutativity	All	186	97.8%	97.8%
Related Fact	All	124	88.7%	95.6%
	Related Fact	79	96.0%	96.0%
	Direction	25	76.0%	86.4%
	No Application	16	62.5%	71.4%
Operand > Product	All	90	90.0%	96.4%
	Great-than-Rule	75	92.0%	95.8%
	Direction	22	63.6%	82.4%
	None	3	66.6%	66.6%
Operand = Product	All	90	88.9%	96.4%
	Multiplication Rule	56	94.6%	96.5%
	Direction	32	75.0%	88.9%
	None	5	60.0%	80.0%

Table 11 continued

Inter-Rater Reliability for Individual Strategies on Conceptual Tasks

Problem Type	Justifications[a]	Number of Observations[b]	Conservative Reliability	Rater Specific Reliability
Estimation	All	90	77.8%	92.1%
	Multiplication	17	82.3%	93.3%
	Ambiguous Justification	33	72.0%	80.0%
	Partial Multiplication	21	57.1%	80.0%
	None	20	40.0%	80.0%
	Addition	8	50.0%	57.0%
Word Problems	All	120	94.2%	98.3%

a "All" refers to all the justifications for the problem type.

b Number of observations in reliability check.

Table 12

Correlations for all Measures for Grade 4 and 6 Children[a]

	Wrat-III	Digit span	Operation span	Median Latency	Percent Retrieval	R.A./ Commutative	Related- Fact	Equal/ Greater	Estimation	Word Problems
Wrat-III	-	.41*	.17	-.35*	.06	.06	.16	.05	.07	.24
Digit span[b]	.28*	-	.26*	-.24	.05	.13	.22	-.07	-.11	.11
Operation span	.33*	.55*	-	-.35*	.31*	.13	.04	-.01	.08	-.03
Median Latency	-.58*	-.22*	-.45*	-	-.31*	-.07	.09	-.08	-.06	-.05
Percent Retrieval	.49*	.31*	.27*	-.59*	-	.03	.10	-.16	.11	-.07
R.A./Commutative[c]	.25*	-.01	-.03	-.01	.16	-	-.01	.17	.23	.03
Related - Fact	.29*	.00	.12	-.21	.11	.29*	-	.05	.08	.36*
Equal/Greater[d]	.47*	.39*	.32*	-.37*	.36*	.20	.25	-	.18	.09
Estimation	.33*	.18	.24	-.27*	.39*	.27	.08	.23	-	-.06
Word Problems	.37*	.55*	.45*	-.26*	.24	.25*	.41*	.15	.13	-

a Correlations are below the diagonal for Grade 4 and above the diagonal for Grade 6. b Backward digit span.

c Repeated-Addition/Commutativity Problems. d Equal-To/Greater Than Problems.

* p < .05.

Table 13

Summary of R^2 change vales for Operation span Regressed on Backward digit span, Solution Latency and Percent Retrieval

Order of Entry of Variables	Grade 4	Grade 6
Model 1		
Backward digit span	.31*	.07
Solution Latency	.11*	.09*
Percent Retrieval	.01	.04
Model 2		
Solution Latency	.20*	.12*
Percent Retrieval	.01	.04
Backward digit span	.21*	.04

*$p < .05$

Table 14

Summary of R^2 change values for WRAT-III Scores Regressed on Backward digit span and Operation span

Order of Entry of Variables	Grade 4	Grade 6
Model 1		
Operation span	.10*	.02
Backward digit span	.02	.16*
Model 2		
Backward digit span	.08*	.17*
Operation span	.04	.01

*$p < .05$

Table 15

Summary of R^2 change values for WRAT-III Scores Regressed on Solution Latency and Proportion Retrieval

Order of Entry of Variables	Grade 4	Grade 6
Model 1		
Solution Latency	.33*	.13*
Proportion Retrieval	.03	.00
Model 2		
Proportion Retrieval	.24*	.00
Solution Latency	.12*	.13*

*$p < .05$

Table 16

Summary of R^2 change values for WRAT-III Scores Regressed on Solution Latency and Operation span/Backward digit span

Order of Entry of Variables	Grade 4	Grade 6
Model 1		
Solution Latency	.33*	.13*
Operation span/Backward digit span	.01	.11*
Model 2		
Operation/Backward digit span	.10*	.17*
Solution Latency	.23*	.07*

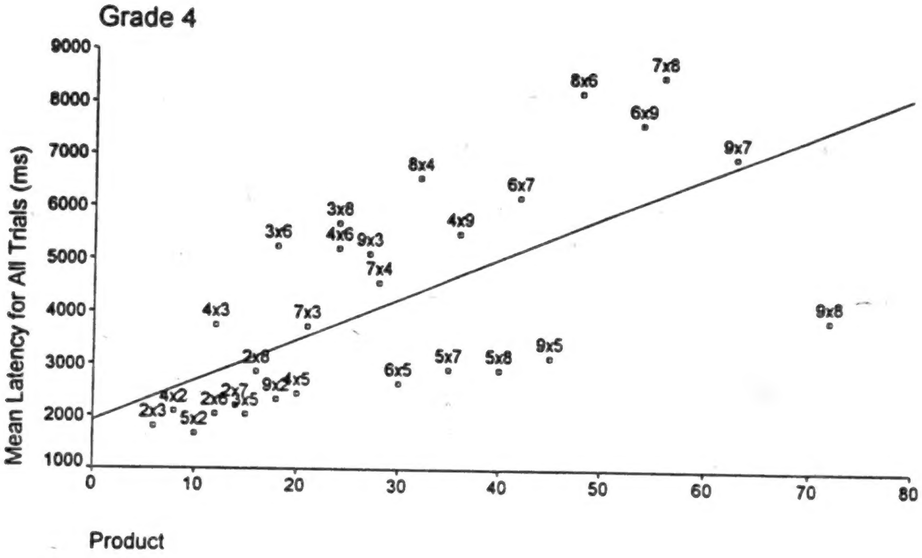
*$p < .01$

Figure Captions

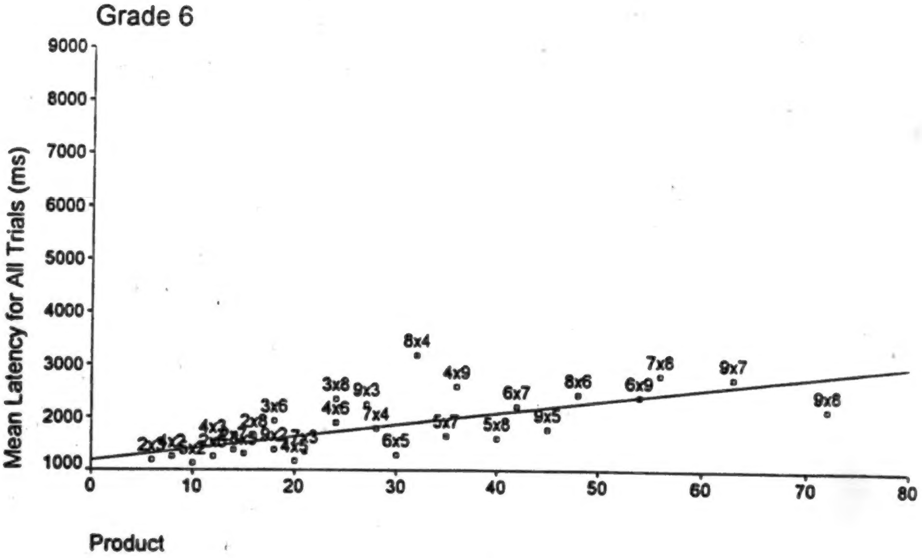
Figure 1 Mean solution latencies for each problem as a function of product size for children in Grade 4 (top panel) and Grade 6 (bottom panel).

Figure 2 Frequency of use of each procedure for Grade 4 (top panel) and Grade 6 (bottom panel) by operand families.

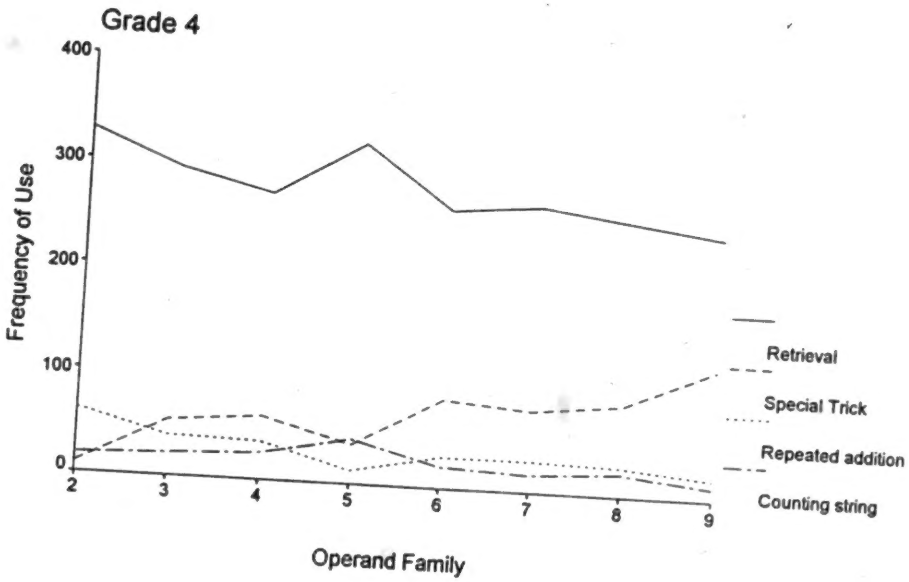
Figure 3 Mean solution times on memory retrieval and all trials for Grade 4 (top panel) and Grade 6 (bottom panel) as a function of product.



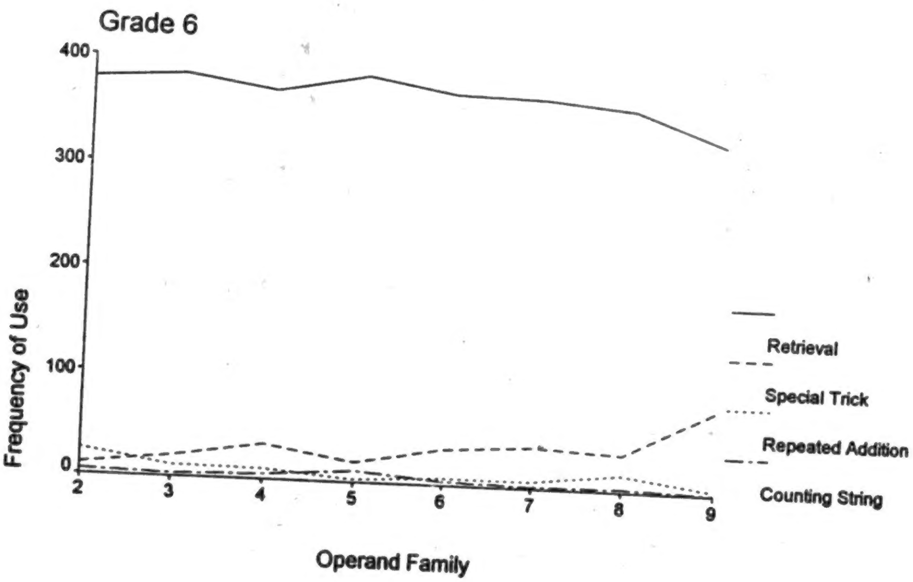
Grade 4



Grade 6

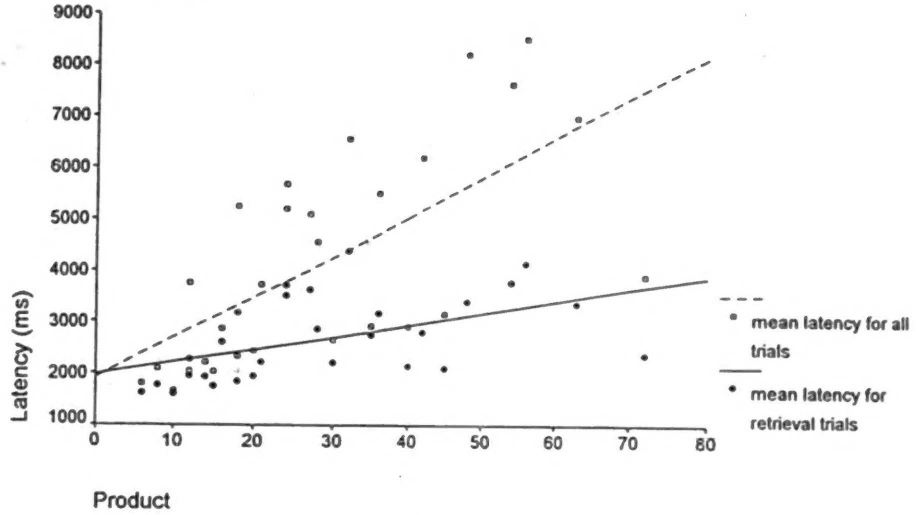


Grade 4

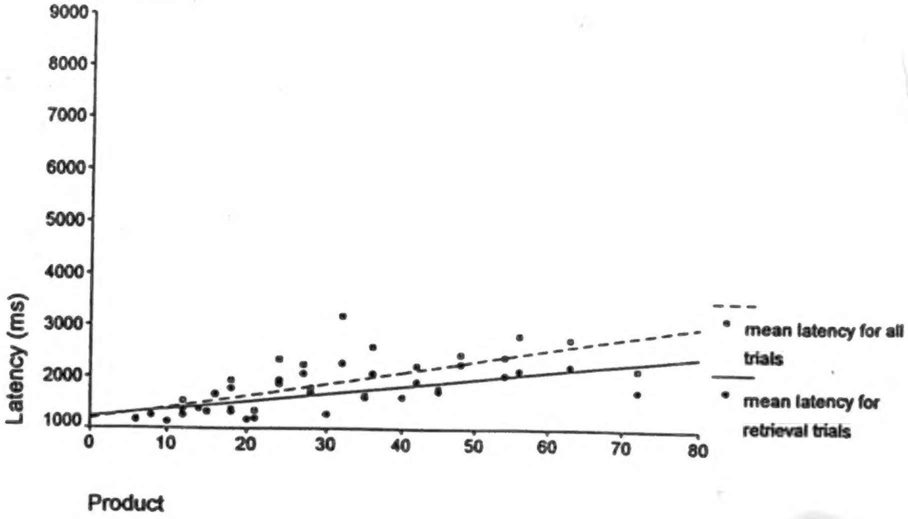


Grade 6

Grade 4



Grade 6



REFERENCES

Ashcraft, M. H. (1982). The development of mental arithmetic: A chronometric approach. Developmental Review, 2, 213-236.

Ashcraft, M. H. (1987). Children's knowledge of simple arithmetic: A developmental model and simulation. In J. Bisanz, C. J. Brainerd, & R. Kail (Eds.), Formal methods in developmental psychology: Progress in cognitive development research (pp. 3O2-338). New York: Springer-Verlag.

Ashcraft, M. H. (1992). Cognitive arithmetic: A review of data and theory. Cognition, 44, 75-106.

Ashcraft, M. H., & Battaglia, J. (1978). Cognitive arithmetic: Evidence for retrieval and decision processes in mental addition. Journal of Experimental Psychology: Human Learning and Memory, 4, 527-538.

Ashcraft, M. H., Fierman, B.A., & Bartolotta, R. (1984). The production and verification tasks in mental addition: An empirical comparison. Developmental Review, 4, 157-170.

Ashcraft, M. H., Donley. R. D., Halas, M. A., & Vakali, M. (1992). Working memory, automaticity, and problem difficulty. In J. I. D. Campbell (Ed.), The nature and origins of mathematical skills (pp 189-253). Amsterdam: Elsevier Science Publishers.

Ashcraft, M. H. & Christy, K. S. (1995). The frequency of arithmetic facts in elementary texts: Addition and multiplication in grades 1-6. Journal for Research in Mathematics Education, 26, 396-421.

Alberta Education (1994). Program of Studies: Elementary Schools.

Baddeley, A. D. (1986). Working memory. New York: Oxford University Press.

Baddeley, A. D., & Hitch, G. (1974). Working memory. In G. Bower (Ed.), The psychology of learning and motivation (vol. 8, pp. 47-89). New York: Academic Press.

Baroody, A. J. (1987). Children's mathematical thinking: A developmental framework for preschool, primary, and special education teachers. New York: Teachers College Press.

Baroody, A. J. (1995). The role of the number-after rule in the invention of computational shortcuts. Cognition and Instruction, 13, 189-219.

Baroody, A. J., & Gannon, K. (1984). The development of the commutativity principle and economical addition strategies. Cognition and Instruction, 1, 321-339.

Baroody, A. J., & Ginsburg, H. P. (1986). The relationship between initial meaningful and mechanical knowledge of arithmetic. In J. Hiebert (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 75-112). Hillsdale, NJ: Erlbaum.

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh & M. Landua (Eds.), Acquisition of mathematical concepts and processes (pp. 57-90). New York: Academic Press.

Bisanz, J., & LeFevre, J. (1990). Strategic and nonstrategic processing in the development of mathematical cognition. In D. F. Bjorklund (Ed.), Children's strategies: Contemporary views of cognitive development (pp. 213-244). Hillsdale, NJ: Lawrence Erlbaum Associates.

Bisanz, J., & LeFevre, J. (1992). Understanding elementary mathematics. In J. Campbell (Ed.), The nature and origins of mathematical skills (pp. 113-136). Elsevier Science Publishers.

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. Developmental Psychology, 27, 777-786.

Campbell, J. I. D. (1994). Architectures for numerical cognition. Cognition, 53, 1-44.

Campbell, J. I. D. (1987). Network interference and mental multiplication. Journal of Experimental Psychology: Learning, Memory, and Cognition, 13, 109-123.

Campbell, J. I. D., & Graham, D. J. (1985). Mental multiplication skill: Structure, process, and acquisition. Canadian Journal of Psychology, 39, 338-366.

Carpenter,. T.P., Corbitt, M.K., Kepner, H.S., Lindquist, M.M., & Reys, R. E. (1980). Solving verbal problems: Results and implications from the National Assessment. Arithmetic Teacher, 28, 8-12.

Carpenter, T. P., Lindquist, C., Brown, C., Kouba, V., Silver, E., & Swafford, J. (1988). Results of the fourth NAEP assessment of mathematics: Trends and conclusions. The Arithmetic Teacher, 4, 38-41.

Carpenter, T. P., Matthews, W., Lindquist, M. M., & Silver, E.A. (1984). Achievement in mathematics: Results from the National Assessment. Elementary School Journal, 84, 485-495.

Case, R., Kurland, D.M., & Goldberg, J. (1982). Operational efficiency and the growth of short-term memory span. Journal of Experimental Child Psychology, 33, 386-404.

Cooney, J. B., & Ladd, S. F. (1992). The influence of verbal protocol methods on children's mental computation. Learning and Individual Differences, 4, 237-257.

Cooney, J. B., Swanson, H. L., & Ladd, S. F. (1988). Acquisition of mental multiplication skill: Evidence for the transition between counting and retrieval strategies. Cognition & Instruction, 5, 323-345.

Daneman, M., & Carpenter, P. A. (1983). Individual differences in integrating information between and within sentences. Journal of Experimental Psychology: Learning, Memory, and Cognition, 9, 561-583.

Dixon, J. A., & Moore, C. F. (1996). The developmental role of intuitive principles in choosing mathematical strategies. Developmental Psychology, 32, 241-253.

Geary, D. C. (1994). Children's mathematical development. Washington, DC: American Psychological Association.

Geary, D.C., Bow-Thomas, C.C., & Yau, Y. (1992). Counting knowledge and skill in cognitive addition: A comparison of normal and mathematically disabled children. Journal of Experimental Child Psychology, 54, 372-391.

Geary, D. C., Brown, S. C., & Samaranayake, V. A. (1991). Cognitive addition: A short longitudinal study of strategy choice and speed-of-processing differences in normal and mathematically disabled children. Developmental Psychology, 27, 787-797.

Goldman, S. R., Pellegrino, J. W., & Mertz, D. L. (1988). Extended practice of basic addition facts: Strategy changes in learning-disabled children. Cognition and Instruction, 5, 223-265.

Graham, D. J., & Campbell, J. I. D. (1992). Network interference and number-fact retrieval: Evidence from children's alphaplication. Canadian Journal of Psychology, 46, 65-91.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. Journal for Research in Mathematics Education, 22, 170-218.

Greeno, J. G., Riley, M. S., & Gelman, R. (1984). Conceptual competence and children's counting. Cognitive Psychology, 16, 94-143.

Hiebert, J., & LeFevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Hitch, G. C. (1978). The role of short-term working memory in mental arithmetic. Cognitive Psychology, 10, 302-323.

Howden, H. (1989). Teaching number sense. The Arithmetic Teacher, 2, 6-11.

Hubbard, K. E. (1993). The effects of problem format and age on the use of derived-fact strategies in arithmetic. Unpublished honor's thesis, University of Alberta, Edmonton, Alberta, Canada.

Inhelder, B., & Piaget, J. (1958). The growth of logical thinking from childhood to adolescence. New York: Basic Books.

Jastak, J. F., & Jastak, S. (1993). Manual for the Wide Range Achievement Test: Third Revision. Wilmington, DE: Jastak Associates, Inc.

Kiel, F. C. (1989). Concepts, kinds, and cognitive development. Cambridge, MA: The MIT Press.

Koshmider, J. W., & Ashcraft, M. H. (1991). The development of children's mental multiplication skills. Journal of Experimental Child Psychology, 51, 53-89.

Krzanowska, E. (1988). Relation between procedural skill and understanding in multiplication. Unpublished master's thesis, University of Alberta, Edmonton, Alberta, Canada.

LeFevre, J. (1993). Individual differences in working memory. Unpublished manuscript.

LeFevre, J., Bisanz, J., Daley, K. E., Buffone, L., Greenham, S. L., & Sadesky, G. S. (1996a). Multiple routes to solution of single-digit multiplication problems. Journal of Experimental Psychology: General.

LeFevre, J., Sadesky, G. S., & Bisanz, J. (1996b). Selection of procedures in mental addition: Reassessing the problem size effect in adults. Journal of Experimental Psychology: Learning, Memory, and Cognition, 22, 216-230.

Lemaire, P., & Siegler, R. S. (1995). Four aspects of strategic change: Contributions to children's learning of multiplication. Journal of Experimental Psychology: General, 124, 83-97.

Loftus, E. F. (1992). When a lie becomes memory's truth: Memory distortion after exposure to misinformation. Current Directions in Psychological Science, 1, 121-123.

Logie, R. H., & Baddeley, A. D. (1987). Cognitive processes in counting. Journal of Experimental Psychology, 13, 310-326.

Logie, R. H., Gilhooly, K. J., & Wynn, V. (1994). Counting on working memory in arithmetic problem solving. Memory and Cognition, 22, 395-410.

McKenzie, W. S. (1993). Dealing with mathematics difficulties in a large board of education: A descriptive profile. Unpublished doctoral thesis, University of Toronto, Toronto, Ontario, Canada.

Miller, K., Perlmutter, M., & Keating, D. (1984). Cognitive arithmetic: Comparison of operations. Journal of Experimental Psychology: Learning, Memory, and Cognition, 10, 46-60.

Nunes, T., & Bryant, P. (1995). Do problem situations influence children's understanding of the commutativity of multiplication? Mathematical Cognition, 1, 245-260.

Ohlsson, S., & Rees, E. (1991). The function of conceptual understanding in the learning of arithmetic procedures. Cognition and Instruction, 8, 103-179.

Piaget, J. (1952). The child's concept of number. New York: W.W. Norton.

Putnam, R. T., deBettencourt, L. U., & Leinhardt, G. (1990). Understanding of derived-fact strategies in addition and subtraction. Cognition and Instruction, 7, 245-285.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), The development of mathematical thinking (pp. 109-151). New York:

Academic Press.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. Journal for Research in Mathematics Education, 20, 8-27.

Rourke, B. P., & Strang, J. D. (1983). Subtypes of reading and arithmetic disabilities: A neuropsychological analysis. In M. Rutter (Ed.), Developmental Neuropsychiatry (pp. 473-488). London: The Guildford Press.

Russo, J. E., Johnson, E. J., & Stephens, D. L. (1989). The validity of verbal protocols. Memory & Cognition, 17, 759-769.

Siegel, L. S., & Ryan, E. B. (1989). The development of working memory in normally achieving and subtypes of learning disabled children. Child Development, 60, 973-980.

Siegler, R. S. (1987). The perils of averaging data over strategies: An example from children's addition. Journal of Experimental Psychology: General, 116, 250-264.

Siegler, R. S. (1988a). Strategy choice procedures and the development of multiplication skill. Journal of Experimental Psychology: General, 117, 258-275.

Siegler, R. S. (1988b). Individual differences in strategy choices: Good children not-so-good children, and perfectionists. Child Development, 59, 833-851.

Siegler, R. S. (1989). Hazards of mental chronometry: An example from children's subtraction. Journal of Educational Psychology, 81, 497-506.

Siegler, R. S. (1991). Children's thinking (2nd ed.). Englewood Cliffs, NJ: Prentice

Hall.

Siegler, R. S., & Lemaire, P. (1997). Older and younger adults' strategy choices in multiplication: Testing predictions of ASCM via the choice/no-choice method. Journal of Experimental Psychology: General, 126, 71 - 91.

Siegler, R. S., & McGilly, K. (1989). Strategy choices in children's time telling. In I. Levin & D. Zakay (Eds.), Time and human cognition: A life-span perspective (pp. 181-218). New York: North-Holland.

Siegler, R. S., & Shipley, E. (1995). Variation, selection, and cognitive change. In G. Halford & T. Simon (Eds.), Developing cognitive competence: New approaches to process modelling (pp. 31-76). Hillsdale, NJ: Lawrence Erlbaum Associates.

Siegler, R. S., & Schrager, J. (1984). Strategy choices in addition and subtraction: How do children know what to do? In C. Saffian (Ed.), Origins of cognitive skills (pp. 229-293). Hillsdale, NJ: Lawrence Erlbaum Associates.

Stazyk, E. H., Ashcraft, M. H., & Hamann, M. S. (1982). A network approach to mental multiplication. Journal of Experimental Psychology: Learning, Memory, and Cognition, 8, 320-335.

Steffe, L., & von Glaserfeld, E. (1983). Children's counting types: Philosophy, theory, and application. New York: Praeger Publishers.

Strang, J. D., & Rourke, B. P. (1985). Adaptive behavior of children who exhibit specific arithmetic disabilities and associated neuropsychological abilities and deficits. In B. P. Rourke (Ed.), Neuropsychology of learning disabilities: Essentials of subtype analysis (pp. 167-183). London: The Guilford Press.

Sweller, J., Mawer, R. F., & Ward, M. R. (1983). Development of expertise in mathematical problem solving. Journal of Experimental Psychology: General, 112, 639-661.

Thompson, C., & Rathmell, E. (1989). By way of introduction. The Arithmetic Teacher, 2, 2-3.

Turner, M. L., & Engle, R. W. (1989). Is working memory capacity task dependent? Journal of Memory and Language, 28, 127-154.

Widaman, K. F., Geary, D. C., Cormier, P., & Little, T. D. (1989). A componential model for mental addition. Journal of Experimental Psychology: Learning, Memory, and Cognition, 15, 898-919.

Widaman, K. F., & Little, T. D. (1992). The development of skill in mental arithmetic: An individual differences perspective. In J. I. D. Campbell (Ed.), The nature and origins of mathematical skills (pp 189-253). Amsterdam: Elsevier Science Publishers.

Appendix A

Instructions to Participants

Wide Range Achievement Test - Revision 3: Arithmetic Sub-test

Directions for Arithmetic: Begin the test with the written computations. In examining children 8 years and up, say: This is an arithmetic test. Turn to pages 2 and 3 where it says Arithmetic). Look at the problems printed below the heavy line and going on to the next page. (Hold test form up and point to both pages.) I'd like to know how many of the problems you can figure out. Look at each problem carefully to see what you are supposed to do - add, subtract, multiply or divide - and then put down your answer in the space on or under the lines. Should you wish to figure on the paper, you may use the empty spaces or the sides to write on. There is also space at the bottom of page 3(indicate by pointing). First do the top row, then the second row, the third, etc. The problems get more difficult as you go down the page and on the next. Don't spend too much time on any one problem. You can skip a problem if it is too difficult for you, but do as many as you can one by one. You will have 10 minutes. Now, go ahead and do as many as you can. Don't forget to go on to the second page of problems.

Computational Task

These are the instructions to precede presenting the multiplication problems on the computer: I am trying to find out how children solve different math problems. Can you help me with this, (child's name)? I would like you to solve a number of multiplication problems and tell me how you solved them. I will show you some problems on the computer screen one at a time, and when you get an answer for each problem, say it out loud. I will then ask you to tell me what you were thinking as you solved the problem. You will see a math problem on the screen. Remember to say your answer out loud and as quickly as you can without making mistakes. Then tell me how you got your answer. For example, you may just know it, you may count in your head, you may have some special trick you use to solve some problems. However you solve a problem, I would like you to tell me about it. Before each problem is shown you will see a star. When you are ready to solve the problem say "Go" and the problem will appear on the screen. (Place microphone on child:) You will say "go" into this microphone, which is connected to the computer. The microphone will also help us to see how long it takes you to solve each problem. (Test microphone.) To give you some practice, I want you to tell me what you think the answer is for the following problems. Some of the problems will be easy and some will be hard. When you are correct you will hear one sound, and when you are incorrect you will hear another. But what I am interested in is how you think about math, so don't worry if you have trouble with some of the problems. This is not a test. All I want you to do is try your best. Can we begin?"

[Start program here and continue until practice problems are finished]

So those were some practice problems. Now lets continue with new problems. Remember, I want you to tell me the answer as quickly as you can without making mistakes. Then tell me what you were thinking as you solved the problem. Are you ready to begin?

[If the child has difficulty solving a problem, it still may be useful to get information on how he or she is trying to solve the problem. Some useful prompts to help a children explain how they are trying to solve a problem are listed below.]

Tell me how you solved this problem? If you could solve the problem, how would you do it?

What did you do first, and then what did you do? What numbers did you look at first, and then what numbers did you look at? Tell me what you are thinking?

Remember you may just know it, you may count in your head, you may have some special trick you use to solve some problems.

[If the child says "I don't know", say] Do you want to guess? How did you come up with that guess? [If the child doesn't want to guess, say] How did you try to get an answer?

[If the child guesses, code "guess" and ask] How did you come up with that guess?

Appendix B

Computational Task Stimuli

4 x 2

6 x 7

3 x 8

9 x 7

2 x 3

4 9

6 x 5

9 x 2

4 x 3

2 x 7

4 x 6

9 x 8

3 x 6

5 x 7

8 x 4

2 x 6

9 x 3

8 x 6

4 x 5

2 x 8

7 x 3

5 x 2

6 x 9

3 x 5

7 x 8

9 x 5

7 x 4

5 x 8

Appendix C

Instructions for Conceptual Tasks

The following instructions were read to children at the beginning of each session: "I have the homework of a boy (girl) in Grade 4 (6). He (she) has answered a number of different multiplication problems. I want you to help me correct each problem without multiplying, O.K.? I want you to look at each problem and tell me if the boy/girl got the right answer. I also want you to tell me why you think that. Are you ready? " Sex and grade were selected to match those of the child. As well, instructions for the specific tasks were also read to the children.

Comparison

The following instructions were read for the comparison task: "Look at this group of problems here (show first page of repeated addition, commutativity, and related-fact problems). For these problem, he (she) was given the correct answer to one problem (point to the problem on the left), and he (she) tried to use it to solve another problem (point to problem on the right). So, for all the problems on the left side (point to problems on left side), the correct answer is given (point to answers on left side). On the right side are the problems that the boy (girl) had to solve (point to problems on right side). Sometimes the correct answer here (point to problem on right side) might not be the same as the answer here (point to left side), and sometimes they might be the same.

So, for this problem (cover all problem pairs except first one and state problem on the right side without answer) he (she) knew from looking at the left side that (state problem on left side with answer). He (she) tried to use the information from the problem on the left side (point to problem on left) to solve the problem on the right side (point to problem on the right) and got this answer (state answer presented for problem on the right). Can you tell without multiplying whether his (her) answer is correct or not? Why (or) Why not?"

If the child gave a response that indicated he or she multiplied the numbers on the right, the following prompt was given: "Well yes that is correct using multiplication, but if you couldn't multiply in your head, could you find out if the answer is correct or not by using information from this problem (point to problem on the left)? Is there a way you can use the information from this problem (point to problem on the left) to see if his (her) answer to this problem (point to problem on the right) is correct or not? Why, why not?"

If the children only examined whether the numbers on the left and right were the same to determine if the answer was correct, the following prompt was given: "Remember, sometimes the correct answer here (point to problem on the right) might not be the same as the answer here (point to problem on the left), and sometimes it might be the same. So, can you tell me another reason why you think that the answer is correct or incorrect?"

For related-fact problems, if the children understood that the problem can be decomposed, and notes the direction of change but did not give a specific answer, the following prompt was given: "Can you tell me how much more this problem is going to be (point at problem on right) than this one (point at problem on left)?"

Number Sense

The following instructions were read for the number sense task: "Here are some more multiplication problems that the boy (girl) worked on. I know you could solve them by multiplying, but I want to find out if you can tell if the answers are probably right or wrong, without multiplying. So, can you tell without multiplying whether his (her) answer is correct or not? Why? (or Why not?)"

If the children solved the problem by multiplying, the following question was asked: "If you couldn't multiply, is there anyway you could estimate, just by looking, whether his (her) answer is correct or not? Why? (or Why not?)"

Word Problems

The following instructions were given for the word problems task: "I want you to read each of the following word problems aloud, and tell me how you would solve the problem. For example you could tell me what numbers you looked at, and what you did with those numbers."

Proofs

The following instructions were given for the proofs task: "Look at this problem, 5 + 4 = 9. I can prove the 5 + 4 = 9 using these beads. Watch me. First I count out 5 beads, then I count out 4 more. When I add them together and count them I find there are 9 beads. I want you to use these beads to for the following multiplication problems."

Appendix D

Stimuli for the Comparison Task

Children were presented with three types of problem on the comparison task.

Repeated Addition Problems

13 + 13 + 13 + 13 + 13 + 13 = 78	13 x 6 = 78
47 + 47 + 47 = 141	3 x 47 = 141
28 + 28 + 28 + 28 = 112	9 x 28 = 112
36 + 36 + 36 + 36 + 36 = 180	36 x 4 = 180
29 + 29 + 29 + 29 = 116	4 x 29 = 116
44 + 44 + 44 = 132	44 x 8 = 132
17 + 17 + 17 + 17 + 17 = 85	6 x 17 = 85
32 + 32 + 32 + 32 + 32 = 160	32 x 5 = 160

Commutativity Problems

4 X 64 = 256	64 x 4 = 236
8 X 59 = 472	59 x 8 = 472
72 X 5 = 360	5 x 72 = 360
51 X 9 = 459	9 x 51 = 469
63 X 7 = 441	7 x 63 = 441
6 X 77 = 462	77 x 6 = 422

Related-Fact Problems

35 X 4 = 140   34 x 4 = 136

37 X 5 = 185   38 x 5 = 205

27 X 8 = 216   26 x 8 = 198

23 X 7 = 161   24 x 7 = 168

Appendix E

Stimuli for the Number Sense Task

Children were presented with three types of problem on the number sense task.

Operand-Equal-to- Product Problems

85 x 5 = 85

32 x 6 = 32

8 x 54 = 54

Operand-Greater-than-Product Problems

5 x 67 = 55

37 x 3 = 31

87 x 4 = 78

Estimation Problems

42 x 3 = 136

4 x 53 = 202

72 x 7 = 514

Appendix F

Stimuli for the Word Problems Task

Children were asked to solve four different types of problems.

Repeated Addition/Irrelevant Information

At Dan's car repair shop there were 10 cars. Each car had 3 flat tires. Tires cost 50 dollars each. How many tires did Joe have to fix?

Insufficient Information

There are 5 shelves for books and 3 shelves for toys in Christine's room. Christine put some books on each shelf. How many books are there in her room?

Multiplicative Compare

A soccer ball costs 24 dollars. John has 5 dollars. Ruth has 3 times as many dollars as John. If John and Ruth put their money together, do they have enough money to buy the soccer ball.

Cartesian Multiplication

Heather has 4 pairs of pants and 3 shirts. How many different pant and shirt outfits can she make?

Appendix G

Stimuli for the Proofs Task

Children were presented with a number of beads and asked the following questions:

"Can you prove to me, using these beads, why 3 x 4 = 12?"

"Can you prove to me, using these beads, why 2 x 9 = 18?"

"Can you prove to me, using these beads, why 6 x 4 = 24?"

Appendix H

Backward digit span Stimuli

Item	Trial 1	Trial 2
1.	2-5	6-3
2.	5-7-4	2-5-9
3.	7-2-9-6	8-4-9-3
4.	4-1-3-5-7	9-7-8-5-2
5.	1-6-5-2-9-8	3-6-7-1-9-4
6.	8-5-9-2-3-4-2	4-5-7-9-2-8-1
7.	6-9-1-6-3-2-5-8	3-1-7-9-5-4-8-2

Appendix I

Operation span Stimuli

Item	Trial 1	Trial 2
1.	4x2, 6x7	9x7, 3x8
2.	2x3, 4x9, 4x3	6x5, 2x7, 9x8
3.	9x2, 8x6, 4x6, 5x7	3x6, 8x4, 5x2, 6x9
4.	4x5, 7x4, 2x6, 5x8, 9x3	7x3, 9x5, 2x8, 7x8, 3x5

Appendix J

Coding Scheme for the Conceptual Tasks

The coding scheme was designed to identify the justifications children used on four different tasks that required conceptual understanding of multiplication. Justifications for each task are coded separately. Within each task there are a number of different problem types. Some justifications are limited to specific problem types, and others may be applied to different kinds of problems.

Comparison

In the comparison task children were asked to look at two adjacent problems. The problem on the left side included the correct answers. The child's task was to determine, based on the information provided in the first problem, if the answer to the problem on the right was correct or not. Problems on the left and right were related by three different principles: repeated addition, commutativity, and related facts.

Repeated Addition Problems

For these problems, the problem on the left side was a correct repeated addition problem (e.g., 28 + 28 + 28 + 28 = 112) and the problem on the right side was a multiplication problem (e.g., 9 x 28 = 112). The child's task was to determine, given the information provided on the left side, if the answer for the problem on the right side was correct or not without multiplying or dividing.

Justifications Specific to Repeated Addition Problems

Repeated Addition Application. The child's response is coded as repeated addition application if he or she demonstrates knowledge of repeated addition when comparing the two problems. Specifically, the child must display the knowledge that if a number is repeatedly added, it yields the same result as multiplying that number by the number of times it was added. As examples, the following responses were coded as repeated addition application.

"They are the same problems, just written differently."
"They have the same number of groups."
"You have n numbers here and it is multiplied by n."
"They are only n groups of 44 here but this problem has q groups of 44."
"If you have the same number added more than once, you times it by the number of times it is written down."

Commutativity Problems

For these problems, the problem on the left side is a correct multiplication problem (e.g., 4 x 64 = 256). The problem on the right side includes the same multiplier and multiplicand as the problem on the left, but in reverse order. The answer given for the problem on the left may be the same, or it may be different for the problem on the right (e.g., 64 x 4 = 236).

Justifications Specific to Commutativity Problems

Commutativity Application. The child's response is coded as commutativity application if he or she demonstrates knowledge of commutativity when comparing the two problems. Specifically, the child must understand that, when multiplying two numbers, it does not matter which number is the multiplicand and which is the multiplier, the answer is still the same. As examples, the following responses were coded as commutativity application.

"They are the same problems, the numbers are just reversed."

"The numbers are just switched around."

"The numbers are just backwards."

"The numbers are just turned around."

Related-Fact Problems

For these problems, the problem on the left side is a correct multiplication problem (e.g., 37 x 5 = 185). The problem on the right side is a related multiplication problem (38 x 5 = 205). Specifically, it is related by the fact that while the multiplier stays the same, the multiplicand either increases or decreases by one.

Justifications Specific to Related-Fact Problems

Related-Fact. The child must demonstrate knowledge of the relations between the two problems, including the magnitude and direction of change. As long as he or she demonstrates knowledge of correct direction and magnitude change it does not matter if his or her addition or subtraction is correct. As examples, the following responses were coded as related-fact application.

"You just add 7 more to the answer because this side has 21 7s and this side has 22 7s."

For the problem 23 x 7 = 161/24 x 7 = 168, children states: "That is correct because 161 + 7 = 168."

Direction. The child must demonstrate some knowledge of the appropriate direction of change from one problem to the next, but he or she does not demonstrate knowledge of the magnitude of the change. As examples, the following responses were coded as direction.

"The left side is going to be more because here it is 23 (points to the left) and here it is 24 (points to the right)." (Experimenter: How much more?") "I don't know."

For the problems 27 x 8 = 216 and 26 x 8 = 198 the child states: "27 is 1 ahead of 26 so answer needs to be smaller."

On the related-fact problem 27 x 8 = 216 and 26 x 8 = 198, they might state that the answer is correct because 27 > 26 and 216 > 198.

Magnitude. The child must demonstrate some knowledge of the magnitude of change between the two problems but are confused or incorrect about whether that magnitude should be added or subtracted to the answer on the left. As examples, the following responses were coded as magnitude.

For the problems 27 x 8 = 216 and 26 x 8 = 198, the child states: "You add 8 because 26 is one less than 27."

Justifications Used on All Problems

The following justifications can apply to all three problem types.

No Application of a Principle. The child shows no evidence of seeing relations between the two problems. Example responses include:

(Experimenter: "Can you tell without multiplying or dividing ...?) child responds, "No I can't".

"They are two different problems."

Unprincipled Arithmetic Comparison. The child uses an arithmetic operation such as addition or subtraction to compare two problems, but does so in an unprincipled manner. Example responses include:

For the related - fact problem 27 x 8 = 216 and 26 x 8 = 198, the child replies that you solve it by doing 27 - 8, to see if it is correct or the child says you add 18 onto 198 to get 216.

Division. The child uses division to compare the two problems. An example response follows:

For the problem 7 x 63 = 441/ 63 x 7 = 441 the child states: "I know it is right because if you divide 7 into 441 you get 63.

Addition. The child uses addition to compare the two problems. This code is not used on repeated addition problems. An example response follows:

For the problem 7 x 63 = 441/ 63 x 7 = 441 the child states: "I know it is right because if you add 63 7 times you get 441."

Computational. The child multiplies the problem on the right together to determine if it is correct or not.

Other. Some other strategy is used to compare the two problems. For example:

On the commutativity problem 6 x 77 = 462/77 x 6 = 422, the child states: "The answer is incorrect because the answers are different", with out explaining why they can't be different.

Number Sense

In the Number sense task the children were asked to determine whether the answers to a number multiplication problems were probably right, or probably wrong, without multiplying or dividing. Problems were designed so that three basic principles of number sense could be used to judge whether the answer to a problem was correct or not: estimation, multiplying by one, and

the allowable minimum magnitude of a product in multiplying two whole numbers.

Estimation Problems

For these problems, the answer given was close to the correct answer (e.g., 72 x 7 = 514). Justifications for these problems are coded according to the degree that children report using multiplication to verify the correctness of the answers.

Justifications Specific to Estimation Problems

Multiplication. The child reports that they multiplied all the digits in the problem together, and provide an answer, to see if the given answer is correct. As examples, the following responses were coded as multiplication.

For the problem 4 x 53 = 202, the child states, "That is incorrect because 3 x 4 is 12, and 4 x 5 is 20, carry the one, so it is 21. The correct answer is 212, not 202. "

For the problem 42 x 3 = 136, the child states "3 x 4 is 12, not 13" or they state that "2 x 3 is 6 and 4 x 3 is 12, not 13."

For the problem 4 x 53 = 202, the child states that "4 x 3 = 12 and 4 x 5 is 20 so the answer can't be right."

Partial Multiplication. Although the child reports multiplying some of the digits in the problem together, it is clear that he or she has not used all the digits in the problem does not provide a final answer. As an example, the following response was coded as partial multiplication.

The child multiplies only some of the digits from the operands to estimate answer. For the problem 4 x 53 = 202 the child states "It looks close because 4 x 50 is 200."

Ambiguous Justification/Magnitude. The child justifies his or her response with some ambiguous justification related to the appropriate magnitude of the answer. The child does not report using multiplication to solve the problem. As examples, the following responses were coded as ambiguous justification/magnitude.

"It looks right because the number would be about that size."

For the problem 72 x 7 = 514, the child states "72 is a high number, therefore the answer would be a high number."

"The answer should be a little less, maybe one or two."

Ambiguous Justification/Direction. The child justifies a response with an ambiguous justification related to the appropriate direction of the operation. As examples, the following responses were coded as ambiguous justification/direction.

"That would be the correct answer because when you multiply these numbers together they go up."

For the problem 72 x 7 = 514, the child states "That could be right because 72 x 7 goes higher."

Ambiguous Justification/Direction and Magnitude. The child justifies his or her response with an ambiguous reference to both the appropriate magnitude of the answer and the correct direction of the operation. As an example, the following response was coded as ambiguous justification/direction and magnitude.

For the problem 4 x 53 = 202, responses: "4 x 5 would be a big number added to the other numbers (4 x 3), it would be the right size."

Ambiguous Justification. The child gives an ambiguous justification for why the answer would be right or wrong. As an example, the following response was coded as ambiguous justification.

"The answer would be around the same number as the one given here" (the child points to the product on the right).

Operand-Equal-to-Product Problems

For these problems the answer was equal to one of the operands and the other operand was not equal to 1 (e.g., 32 x 6 = 32).

Justifications for Operand-Equal-to- Product Problems

Multiply by 1 Rule. The child demonstrates knowledge of the logical principle that the product can only equal one of the operands if the other operand is equal to one. As an example, the following response was coded as multiply by one rule.

For the problem 54 x 8 = 54 the child states "That is wrong because only 1 times 54 is 54. The child may also state "their are 8 groups of 54 not 1 group of 54."

Direction. The child demonstrates the knowledge that when multiplying whole numbers the answer is greater than either the multiplicand or the multiplier. As examples, the following responses were coded as direction.

For problem 32 x 6 = 32, the child responds, "if you multiply 32 x 6 you get a lot more than 32, multiplying by 6 makes it bigger." A child may also say "32 wasn't ever multiplied."

For the problem 54 x 8 = 54, the child states "You didn't times it 8 times because 54 is in the question and the answer."

Direction and Magnitude. The child uses the knowledge that when multiplying two whole numbers the answer must be greater than either of the numbers in the problem, and they indicate the approximate magnitude the answer must be. As an example, the following response was coded as direction and magnitude.

"A two digit number times a 1 digit number will be in the hundreds."

For the problem 8 x 54 = 54, the child states "8 x 4 is 32, add 3 onto other in it would be in the hundreds."

Operand-Greater-than-Product Problems

For these problems, the answer was less than one of the numbers in the problem (e.g., 87 x 4 = 78).

Justifications for the Operand-Greater-than- Product Problems

Greater than Rule. The child demonstrates knowledge that an answer cannot be less than either multiplicand or the multiplier. As an example, the following response was coded as greater-than-rule.

"The answer is less than a number in the question." (e.g., for problem 5 x 67 = 55 the child states 55 is less than 67 or 67 is greater than 55.)

Direction. The child demonstrates the knowledge that when multiplying whole numbers the answer is greater than either the multiplicand or the multiplier. As examples, the following responses were coded as direction.

"If you counted 37 three times it would be more than 31."

For the problem 87 x 4 = 78, the child states that "4 groups of 87 is more than 78."

Direction and Magnitude. The child uses the knowledge that when multiplying two whole numbers the answer must be greater than either of the numbers in the problem, and the child indicates the approximate magnitude the answer must be. As an example, the following response was coded as direction and magnitude.

"A two digit number times a 1 digit number will be in the hundreds."

Partial Multiplication. The child multiplies some, but not all, of the numbers in the problem together to justify their answer. As an example, the following response was coded as partial multiplication.

For the problem 5 x 67 = 55, the child states "That is incorrect because 5 x 7 is already 35."

All Problems

The following justifications can be applied to all of the number sense problems

Other. The child uses some other, ambiguous justification to evaluate problems.

None. The child uses no justification, or an irrelevant, or incorrect justification to evaluate the problem. As examples, the following responses were coded as none.

"It kinda looks right."

"I am not sure how to explain why I think that."

For the problem 8 x 54 = 54, the child states, "Only 0 x 54 is 54."

"I just guessed."

For the problem 4 x 53 = 202, the child states, "4 x 53 is not high enough to equal 202."

Addition. The child uses addition to evaluate the correctness of the answer. As an example, the following response was coded as addition.

For the problem 4 x 53 = 202, the child responds "53 + 53 + 53 + 53 is not 202."

Division. The child uses division to evaluate the answer. As an example, the following response was coded as division.

For the problem 42 x 3 = 136, the child responds, "3 divided by 136 is not 42."

Word Problems

The four word problems were designed to assess children's ability to apply the principles of multiplication in novel, non-routine tasks. The children did not need carry out the computation steps in this task. Rather, they were only required to report on how they would solve each problem.

Repeated Addition/Irrelevant Information

For this problem children are required to apply the principle of multiplication as repeated addition to determine that they need to multiply the first two numbers together.

They also need to realize that the third number is not needed to solve the problem.

Justifications Specific to Repeated Addition Problems

Correct Multiplication. The child appropriately applies multiplication to solve word problem, without using the irrelevant information. As an example, the following response was coded as correct multiplication.

"You multiply the 3 and 10 together to get the answer." (Do you do anything else?) "No". The child may add, "You don't need the 50."

Irrelevant Computation. The child may or may not say to multiply the 3 and the 10 together, but he or she does include the third, irrelevant number in any solution justification reported. As examples, the following responses were coded as irrelevant computation.

"You multiply the 3 by the 10, and then you multiply that by 50."

"You multiply the 3 by the 10, and then you add the 50."

The child correctly multiplies 3 and 10, and also says: "you need to multiply 3 times 50 to find out how much it cost to fix the tires."

Insufficient Information

This problem does not include the information required to solve it.

Justifications Specific to Insufficient Problems

Inappropriate Multiplication. The child reports that to solve the problem the two numbers given in the problem need to be multiplied together. As an example, the following response was coded as inappropriate multiplication.

"You multiply the 3 and the 5 together to get the answer."

Insufficient. The child realizes that the problem does not contain the information needed to solve it. As an example, the following response was coded as insufficient.

"You can't solve this one, because it does not tell how many books where on each shelf."

Unsure. The child realizes that the information provided is insufficient and report that they can't solve it. However they also report that the problem may be solved through a method that they are unfamiliar with. As an example, the following response was coded as unsure.

"I can't solve it." (Why?) "Because it doesn't tell how many books there where on each shelf." (Do you need that information to solve it?) "No, you could probably solve it another way, I just don't know how to."

Multiplicative Compare

This problem involves three steps. To solve it children need to identify what numbers they need to multiply, identify what numbers they need to add, and realize they need to compare

the answer with the number given in the problem.

Justifications Specific to Multiplicative Compare

Complete Solution. The child correctly applies the multiplication, addition, and comparison steps. As examples, the following responses were coded as complete solution.

"You multiply 3 times 5, then add John's 5, to get twenty. No they don't have enough to buy the soccer ball."

"You would multiply the 3 and the 5, add another 5 to see if they have enough."

Incomplete Solution/Addition. The child correctly applies multiplication and addition to solve the problem but fails report that you need to compare the final answer with information given in the problem. As an example, the following response was coded as incomplete solution/addition.

"You add 5 and 15 together." (Where did you get the 15?) "It says 3 times as many as John, and that is 15. You add them together to get your answer"

Incomplete Solution/Compare. The child correctly applies the multiplication and comparison steps, but they fails to carry out the addition step. As an example, the following response was coded as incomplete solution/compare.

"You multiply 3 times 5 and get 15, so they don't have enough money because they need 24 dollars."

Incomplete Solution/Extra. The child correctly applies some but not all of the steps necessary for solving the problem. However, the child also includes an extra, irrelevant step to their solution justification. As an example, the following response was coded as incomplete solution/extra.

"3 X 5 is 15, and then you go 3 X 2 is 6, and 15 + 6 is 21, so they don't have enough money." (Why do you multiply 3 X 2?) "Because it says they had 3 times as many."

Cartesian Multiplication

This problem requires the children to group sets together.

Justifications Specific to Cartesian Problem

Correct Multiplication. The child appropriately applies multiplication to solve word problem. As an example, the following response was coded as correct multiplication.

"You multiply 3 and 4 together."

Incorrect Matching. For the Cartesian problem, the child matches the shirts and pants on a 1 to 1 correspondence. As an example, the following response was coded as incorrect matching.

"She will have 3 outfits, because three pants go with three shirts, and one pair of pants is left over."

"You match three pants with three shirts, and then you use a shirt again with the fourth pair of pants."

Correct Diagram. The child uses a diagram to demonstrate how the clothes can be group into sets or matched appropriately. As an example, the following response was coded as correct diagram.

The child draws, or uses some concrete object to represent the 4 pairs of pants and 3 shirts. He or she then draw lines between the pants and shirts or use the beads to find all the different pant and shirt combinations.

Incorrect Diagram. The child draws, or uses some concrete object to represent the 4 pairs of pants and 3 shirts. However, the child does not accurately demonstrate how this diagram will led to the correct answers.

All Problems

Counting. The child counts fingers or reports counting in his or her head to solve the problem. As an example, the following response was coded as counting.

For the repeated addition problem, the child states, "I used my fingers to count the number of tires to fix, and I got 33."

Addition. The child adds together the numbers in the problem in order to solve it. As an example, the following response was coded as addition.

For the repeated addition problem, the child states "You add the 3 and the 10 together."

For the Cartesian problem, the child states "You add the 3 and the 4 together."

Other. The child uses some other, ambiguous strategy to solve problems. As an example, the following response was coded as other.

"You multiply 3 times 5 to get 15, then you add 24 + 5 + 3, so he can get change."

Proofs

In this task children were asked to prove using bead why the answer for three specific multiplication problems was correct (e.g., 3 x 4 = 12).

Justifications for Proofs

Grouping. The child correctly divides the beads into groups to demonstrates why the answer is correct. As an example, the following response was coded as grouping.

For the problem 3 x 4 = 12, the child divides the beads into 3 groups of 4 or 4 groups of 3.

Inappropriate Grouping. The child incorrectly divides the beads into groups and is unable to prove why the answers given is correct or not. As an example, the following response was coded as inappropriate grouping.

For the problem 3 x 4 = 12, the child divides the beads into a group of 3 and a group of 4, he or she then states, "see a group of three and a group of four is 12."

Other. The child uses some other, ambiguous strategy to try to prove that the answer is correct.